

# Astrophysics

## Lecture 07

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# Four-velocity

The (infinitesimally small) difference between the coordinates of two events is also a four-vector. Dividing by the proper time yields a four-vector, the four-velocity:

$$U^\mu \equiv \frac{dx^\mu}{d\tau} \rightarrow U^0 = \frac{cdt}{d\tau} = c\gamma_u \quad \text{or} \quad \boxed{\vec{U} = \gamma_u \begin{pmatrix} c \\ \mathbf{u} \end{pmatrix}} \quad \text{where} \quad \gamma_u \equiv (1 - u^2/c^2)^{-1/2}$$

$$U^i = \frac{dx^i}{d\tau} = \gamma_u u^i \quad u \equiv \left| \frac{d\mathbf{x}}{dt} \right|$$

$d\tau = dt/\gamma$

length of the four-velocity :

$$\boxed{\vec{U} \cdot \vec{U} = U^\mu U_\mu = -(\gamma_u c)^2 + (\gamma_u \mathbf{u})^2 = -c^2}$$

Transformation of the four-velocity:

The first two equations become:

$$\begin{aligned} U'^0 &= \gamma (U^0 - \beta U^1) & \gamma_{u'} c &= \gamma (c\gamma_u - \beta\gamma_u u^1) \\ U'^1 &= \gamma (-\beta U^0 + U^1) & \gamma_{u'} u'^1 &= \gamma (-\beta c\gamma_u + \gamma_u u^1) \\ U'^2 &= U^2 & \gamma_{u'} u'^2 &= \gamma_u u^2 \\ U'^3 &= U^3 & \gamma_{u'} u'^3 &= \gamma_u u^3 \end{aligned}$$

$$\begin{aligned} \gamma_{u'} &= \gamma\gamma_u (1 - vu'/c^2) \\ \gamma_{u'} u'^1 &= \gamma\gamma_u (u^1 - v) \end{aligned}$$

Note:  $\gamma$  denotes the factor for the relative velocity between two frames.  $\gamma_u$  and  $\gamma_{u'}$  are the factors for a velocity vector measured in  $K$  and  $K'$ , respectively.

velocity component:



speed:

$$\boxed{\begin{aligned} u'^1 &= \frac{u^1 - v}{1 - vu^1/c^2} \\ \gamma_{u'} &= \gamma\gamma_u \left( 1 - \frac{vu^1}{c^2} \right) \end{aligned}}$$

This is the previously derived formula.

This is the transform for speed.

Here,  $u^1 = u \cos \theta$  and  $u'^1 = u' \cos \theta'$

# Momentum and Energy

- Four-momentum of a particle with a mass  $m_0$  is defined by

$$P^\mu \equiv m_0 U^\mu \quad \begin{aligned} P^0 &= m_0 c \gamma_v \\ P^i &= \gamma_v m_0 \mathbf{v} \end{aligned}$$

- In the nonrelativistic limit,

$$P^0 c = m_0 c^2 \gamma = m_0 c^2 \left( 1 - \frac{v^2}{c^2} \right)^{-1/2} = m_0 c^2 + \frac{1}{2} m_0 v^2 + \dots$$

Therefore, we interpret  $E \equiv P^0 c = \gamma_v m_0 c^2$  as the total energy of the particle.

The quantity  $m_0 c^2$  is interpreted as the rest energy of the particle.

Then,  $\mathbf{p} \equiv \gamma_v m_0 \mathbf{v}, \quad P^\mu = (E/c, \mathbf{p})$  Here,  $\mathbf{p}$  is the spatial component of the four-momentum.

Since  $\vec{U}^2 = -c^2$ , we obtain  $\vec{P}^2 = -m_0^2 c^2$ . Comparing with  $\vec{P}^2 = -\frac{E^2}{c^2} + |\mathbf{p}|^2$ , we obtain

$$E^2 = m_0^2 c^4 + c^2 |\mathbf{p}|^2$$

- Photons are massless, but we can still define

$$P^\mu = (E/c, \mathbf{p}), \quad E = |\mathbf{p}| c \quad \rightarrow \quad \vec{P}^2 = 0 \quad \text{for photons}$$

# Wavenumber vector and frequency

- Quantum relations:

$$\begin{aligned} E &= h\nu = \hbar\omega \\ p &= E/c = \hbar k \end{aligned} \quad \left( \begin{array}{l} \omega = 2\pi\nu \\ k = 2\pi/\lambda \end{array} \right)$$

We can define four wavenumber vector:

$$\vec{k} = \frac{1}{\hbar} \vec{P} = \left( \frac{\omega}{c}, \mathbf{k} \right)$$

Note that it's a null vector:

$$\vec{k} \cdot \vec{k} = |\mathbf{k}|^2 - \omega^2/c^2 = 0$$

Then, we obtain an invariant:

$$\vec{k} \cdot \vec{x} = k_\mu x^\mu = \mathbf{k} \cdot \mathbf{x} - \omega t$$

Therefore, **the phase of the plane wave is an invariant.**

$$e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} = e^{i(\mathbf{k}' \cdot \mathbf{x}' - \omega' t')}$$

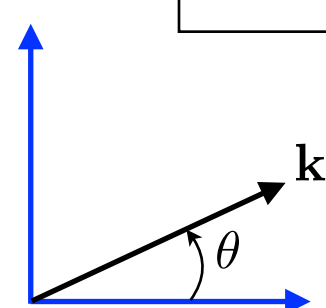
- Transform of  $\vec{k}$  gives the Doppler shift formula.

$$k'^0 = \gamma (k^0 - \beta k^1)$$

$$k'^1 = \gamma (-\beta k^0 + k^1)$$

$$k'^2 = k^2$$

$$k'^3 = k^3$$



$$\omega' = \gamma (\omega - \beta c k^1) = \omega \gamma \left( 1 - \frac{v}{c} \cos \theta \right)$$

$$k^1 = (\omega/c) \cos \theta$$

# \* Tensor Analysis \*

- Definition:

zeroth-rank tensor : Lorentz invariant (scalar)

first-rank tensor : four-vector

second-rank tensor:

$$s' = s$$

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$$

$$T'^{\mu\nu} = \Lambda^{\mu}_{\sigma} \Lambda^{\nu}_{\tau} T^{\sigma\tau}$$

- Covariant components and mixed components:

$$T_{\mu\nu} = \eta_{\mu\sigma} \eta_{\nu\tau} T^{\sigma\tau} \quad T^{\mu}_{\nu} = \eta_{\nu\tau} T^{\mu\tau} \quad T_{\mu}^{\nu} = \eta_{\mu\sigma} T^{\sigma\nu}$$

- Transformation rules:

$$\begin{aligned} T'_{\mu\nu} &= \eta_{\mu\alpha} \eta_{\nu\beta} T'^{\alpha\beta} \\ &= \eta_{\mu\alpha} \eta_{\nu\beta} \Lambda^{\alpha}_{\gamma} \Lambda^{\beta}_{\delta} T^{\gamma\delta} \\ &= \eta_{\mu\alpha} \eta_{\nu\beta} \Lambda^{\alpha}_{\gamma} \Lambda^{\beta}_{\delta} \eta^{\gamma\sigma} \eta^{\delta\tau} T_{\sigma\tau} \\ &= \tilde{\Lambda}_{\mu}^{\sigma} \tilde{\Lambda}_{\nu}^{\tau} T_{\sigma\tau} \end{aligned}$$

$$\begin{aligned} T'^{\mu}_{\nu} &= \eta_{\nu\alpha} T'^{\mu\alpha} \\ &= \eta_{\nu\alpha} \Lambda^{\mu}_{\sigma} \Lambda^{\alpha}_{\delta} T^{\sigma\delta} \\ &= \eta_{\nu\alpha} \Lambda^{\mu}_{\sigma} \Lambda^{\alpha}_{\delta} \eta^{\delta\tau} T^{\sigma}_{\tau} \\ &= \Lambda^{\mu}_{\sigma} \tilde{\Lambda}_{\nu}^{\tau} T^{\sigma}_{\tau} \end{aligned}$$

$$\begin{aligned} T'_{\mu}^{\nu} &= \eta_{\mu\alpha} T'^{\alpha\nu} \\ &= \eta_{\mu\alpha} \Lambda^{\alpha}_{\beta} \Lambda^{\nu}_{\tau} T^{\beta\tau} \\ &= \eta_{\mu\alpha} \Lambda^{\alpha}_{\beta} \Lambda^{\nu}_{\tau} \eta^{\beta\sigma} T_{\sigma}^{\tau} \\ &= \tilde{\Lambda}_{\mu}^{\beta} \Lambda^{\nu}_{\tau} T_{\sigma}^{\tau} \end{aligned}$$

- Symmetric tensor = a tensor that is invariant under a permutation of its indices.

$$T^{\mu\nu} = T^{\nu\mu}$$

- Antisymmetric tensor : if it alternates sign when any two indices of the subset are interchanged.

$$T^{\mu\nu} = -T^{\nu\mu}$$

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- Examples of the second-rank tensors

A product of two vectors:  $A^\mu B^\nu$

$$A'^\mu B'^\nu = \Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau A^\sigma B^\tau$$

The Minkowski metric:

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The Kronecker-delta:

$$\delta^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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- Higher-rank tensors

- Addition:  $A^\mu + B^\mu$ ,  $F^{\mu\nu} + G^{\mu\nu}$
- Multiplication:  $A^\mu B^\nu$ ,  $F^{\mu\nu} G_{\sigma\tau}$
- Raising and Lowering Indices: The metric can be used to change contravariant indices into covariant ones, and vice versa, by the processes of raising and lowering.
- Contraction:  $A^\mu B_\nu \rightarrow A^\mu B_\mu$  scalar  
 $T^{\mu\nu}_\sigma \rightarrow T^{\mu\nu}_\nu$  vector

$$T'^{\mu\nu}_\nu = \Lambda^\mu_\alpha \Lambda^\nu_\beta \tilde{\Lambda}_\nu^\tau T^{\alpha\beta}_\tau = \Lambda^\mu_\alpha \delta^\tau_\beta T^{\alpha\beta}_\tau = \Lambda^\mu_\alpha T^{\alpha\beta}_\beta$$

- Gradients of Tensor Fields: A tensor field is a tensor that is a function of the spacetime coordinates in Cartesian coordinate systems. The gradient operation  $\partial/\partial x^\mu \equiv \partial_\mu$  acting on such a field produces a tensor field of on higher rank with  $\mu$  as a new covariant index.

$$\lambda \rightarrow \frac{\partial \lambda}{\partial x^\mu} \equiv \partial_\mu \lambda \equiv \lambda_{,\mu} \quad \text{vector (gradient)} \quad A^\mu \rightarrow \frac{\partial A^\mu}{\partial x^\mu} \equiv \partial_\mu A^\mu \equiv A^\mu_{,\mu} \quad \text{scalar (divergence)}$$

- **Invariance of form or Lorentz covariance or covariance:** A fundamental property of a tensor equation is that if it is true in one Lorentz frame, then it is true in all Lorentz frames. Covariance plays a powerful role in helping decide what the proper equations of physics are.

# [Covariance of Electromagnetic Phenomena]

- Equation of charge conservation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

The above equation can be written as a tensor equation,

$$\boxed{\frac{\partial j^\mu}{\partial x^\mu} = 0,} \quad j^\mu_{,\mu} = 0 \quad \text{or} \quad \partial_\mu j^\mu = 0$$

$$\begin{aligned} \partial_\mu &\equiv \frac{\partial}{\partial x^\mu} = \left( \frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \\ \partial^\mu &\equiv \frac{\partial}{\partial x_\mu} = \left( -\frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \end{aligned}$$

if the **four-current** is defined by

$$j^\mu = \begin{pmatrix} \rho c \\ \mathbf{j} \end{pmatrix} \quad j_\mu = \begin{pmatrix} -\rho c \\ \mathbf{j} \end{pmatrix}$$

- Note that the Jacobian (determinant) of the transformation from  $x_\mu$  to  $x'_\mu$  is simply the determinant of  $\Lambda$ , which is unity. Therefore, the **four-volume element is an invariant**.

$$dx'_0 dx'_1 dx'_2 dx'_3 = \det \Lambda dx_0 dx_1 dx_2 dx_3 = dx_0 dx_1 dx_2 dx_3$$

Since  $\rho$  is the zeroth component of the four-current, **the charge element within a three-volume element is an invariant**.

$$de = \rho dx_1 dx_2 dx_3$$

$$de' = de$$

It is also an empirical fact that  $e$  is invariant.



- The set of vector and scalar wave equations in the Lorentz gauge is

$$\begin{aligned}\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\frac{4\pi}{c} \mathbf{j} \\ \nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} &= -\frac{4\pi}{c} (\rho c)\end{aligned}$$

If we define the **four-potential**

$$A^\mu = \begin{pmatrix} \phi \\ \mathbf{A} \end{pmatrix}, \quad A_\mu = \begin{pmatrix} -\phi \\ \mathbf{A} \end{pmatrix},$$

then the wave equations can be written as the tensor equations

$$\frac{\partial^2 A^\mu}{\partial x^\nu \partial x_\nu} = -\frac{4\pi}{c} j^\mu, \quad \partial_\nu \partial^\nu A^\mu = -\frac{4\pi}{c} j^\mu, \quad A^{\mu,\nu}_{,\nu} = -\frac{4\pi}{c} j^\mu$$

d'Alembertian operator:  $\square \equiv \frac{\partial^2}{\partial x^\nu \partial x_\nu} \rightarrow \boxed{\square A^\mu = -\frac{4\pi}{c} j^\mu}$

- The Lorentz gauge should be preserved under Lorentz transformations.

$$\nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0 \rightarrow \boxed{\frac{\partial A^\mu}{\partial x^\mu} = 0, \quad \partial_\mu A^\mu = 0,} \quad \text{or} \quad A^\mu_{,\mu} = 0$$

- **Electromagnetic field tensor:**

The fields are expressed in terms of the potentials as

$$\begin{aligned}\mathbf{E} &= -\nabla\phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} &= \nabla \times \mathbf{A}\end{aligned}$$

The  $x$  components of the electric and magnetic fields are explicitly

$$\begin{aligned}E_x &= -\frac{1}{c} \frac{\partial A_x}{\partial t} - \frac{\partial \phi}{\partial x} = \partial^0 A^1 - \partial^1 A^0 \\ B_x &= \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \partial^2 A^3 - \partial^3 A^2\end{aligned}$$

These equations imply that the electric and magnetic fields, six components in all, are the elements of a **second-rank, antisymmetric field-strength tensor**, because a rank two antisymmetric tensor has exactly six independent components.

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \longrightarrow \begin{aligned}F^{0i} &= E_i \\ F^{i0} &= -E_i \\ F^{12} &= -F^{21} = B_3, \dots\end{aligned}$$

covariant field-strength tensor

$$F_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}F^{\alpha\beta}$$



$$F_{0i} = \eta_{0\alpha}\eta_{i\beta}F^{\alpha\beta} = -F^{0i}$$

$$F_{i0} = \eta_{i\alpha}\eta_{0\beta}F^{\alpha\beta} = -F^{i0}$$

$$F_{ij} = \eta_{i\alpha}\eta_{j\beta}F^{\alpha\beta} = F^{ij}$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$



$$F_{0i} = -E_i$$

$$F_{i0} = E_i$$

$$F_{12} = -F_{21} = B_3, \dots$$

- The two Maxwell equations containing sources (inhomogeneous equations):

$$\begin{aligned} \nabla \cdot \mathbf{E} = 4\pi\rho & \qquad \sum_{i=1}^3 \partial_i E_i = \frac{4\pi}{c} j^0 & \qquad -\sum_{i=1}^3 \partial_i F^{i0} = \frac{4\pi}{c} j^0 \\ \nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c} \mathbf{j} & \longrightarrow \partial_2 B_3 - \partial_3 B_2 - \partial_0 E_1 = \frac{4\pi}{c} j^1 & \longrightarrow -\partial_0 F^{01} - \partial_2 F^{21} - \partial_3 F^{31} = \frac{4\pi}{c} j^1 \end{aligned}$$

$$\partial_\mu F^{\mu\nu} = -\frac{4\pi}{c} j^\nu \quad \text{or} \quad \partial_\nu F^{\mu\nu} = \frac{4\pi}{c} j^\mu$$

$$\partial^\mu F_{\mu\nu} = -\frac{4\pi}{c} j_\nu \quad \text{or} \quad \partial^\nu F_{\mu\nu} = \frac{4\pi}{c} j_\mu$$

- The conservation of charge easily follows from the above equation and the asymmetric property.

$$\partial_\mu \partial_\nu F^{\mu\nu} = \partial_\nu \partial_\mu (-F^{\nu\mu}) = -\partial_\mu \partial_\nu F^{\mu\nu} \quad \therefore \quad \begin{array}{l} \partial_\mu \partial_\nu F^{\mu\nu} = 0 \\ \partial_\nu j^\nu = -\frac{c}{4\pi} \partial_\nu \partial_\mu F^{\mu\nu} = 0 \end{array}$$

↑  
index exchange

- The “internal” Maxwell equations (homogeneous equations):

$$\begin{array}{l} \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \end{array} \longrightarrow \begin{array}{l} \sum_{i=1}^3 \partial_i B_i = 0 \\ \partial_2 E_3 - \partial_3 E_2 + \partial_0 B_1 = 0 \end{array} \longrightarrow \begin{array}{l} \partial_1 F^{23} + \partial_2 F^{31} + \partial_3 F^{12} = 0 \\ \partial_2 F^{30} + \partial_3 F^{20} + \partial_0 F^{23} = 0 \end{array}$$

$$\partial_\mu F^{\nu\sigma} + \partial_\nu F^{\sigma\mu} + \partial_\sigma F^{\mu\nu} = 0 \quad \text{or} \quad \partial^\mu F_{\nu\sigma} + \partial^\nu F_{\sigma\mu} + \partial^\sigma F_{\mu\nu} = 0$$

The equation can be written concisely as  $F^{[\mu\nu,\sigma]} = 0$  or  $F_{[\mu\nu,\sigma]} = 0$ , where  $[ ]$  around indices denote all permutations of indices, with even permutation contributing with a positive sign and odd permutation with a negative sign, for example,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = A_{[\nu,\mu]}$$

# - Transformation of Electromagnetic Fields

- Since  $F^{\mu\nu}$  is a second-rank tensor, its components transform in the usual way:

$$F'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} F^{\alpha\beta} = \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} F^{\alpha\beta}$$

For a pure boost along the  $x$ -axis:

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{aligned} E'_x &= F'^{01} = \Lambda^0_0 \Lambda^1_1 F^{01} + \Lambda^0_1 \Lambda^1_0 F^{10} = \gamma^2 E_x - \beta^2 \gamma^2 E_x = E_x \\ E'_y &= F'^{02} = \Lambda^0_0 \Lambda^2_2 F^{02} + \Lambda^0_1 \Lambda^2_2 F^{12} = \gamma E_y - \beta\gamma B_z \\ E'_z &= F'^{03} = \Lambda^0_0 \Lambda^3_3 F^{03} + \Lambda^0_1 \Lambda^3_3 F^{13} = \gamma E_z + \beta\gamma B_y \\ B'_x &= F'^{23} = \Lambda^2_2 \Lambda^3_3 F^{23} = B_x \\ B'_y &= F'^{31} = \Lambda^3_3 (\Lambda^1_0 F^{30} - \Lambda^1_1 F^{31}) = \beta\gamma E_z + \gamma B_y \\ B'_z &= F'^{12} = \Lambda^1_0 \Lambda^2_2 F^{03} + \Lambda^1_1 \Lambda^2_2 F^{12} = -\beta\gamma E_y + \gamma B_z \end{aligned}$$

- In general,

$$\begin{aligned} \mathbf{E}'_{\parallel} &= \mathbf{E}_{\parallel} & \mathbf{B}'_{\parallel} &= \mathbf{B}_{\parallel} \\ \mathbf{E}'_{\perp} &= \gamma (\mathbf{E}_{\perp} + \boldsymbol{\beta} \times \mathbf{B}) & \mathbf{B}'_{\perp} &= \gamma (\mathbf{B}_{\perp} - \boldsymbol{\beta} \times \mathbf{E}) \end{aligned}$$

The concept of a pure electric or pure magnetic is not Lorentz invariant.

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- Lorenz invariants:

- The dot product of  $F$  with itself or “square” of  $F$  is

$$\begin{aligned}
 F^{\mu\nu} F_{\mu\nu} &= \sum_{i=1}^3 F^{0i} F_{0i} + \sum_{i=1}^3 F^{i0} F_{i0} + \sum_{i \neq j} F^{ij} F_{ij} \\
 &= 2 (\mathbf{B}^2 - \mathbf{E}^2)
 \end{aligned}$$

Therefore,  $\mathbf{B}^2 - \mathbf{E}^2$  is invariant under Lorentz transformations.

- The determinant of  $F$  is invariant.

$$\det F = (\mathbf{E} \cdot \mathbf{B})^2$$

Thus  $\mathbf{E} \cdot \mathbf{B}$  is also an invariant.

- The determinant of any second-rank tensor is scalar, since

$$\begin{aligned}
 \det A_{\alpha\beta} &= \det \tilde{\Lambda}_{\alpha}^{\mu} \tilde{\Lambda}_{\beta}^{\nu} A'_{\mu\nu} \\
 &= \left( \det \tilde{\Lambda} \right)^2 \det A'_{\mu\nu} \\
 &= \det A'_{\mu\nu}
 \end{aligned}$$

# [Relativistic Mechanics and the Lorentz Four-Force]

- We can define a **four-acceleration**  $a^\mu$  in exactly the same way as we obtained the four-velocity.

$$a^\mu \equiv \frac{dU^\mu}{d\tau}$$

Note that the four-acceleration and four-velocity are orthogonal:

$$\vec{a} \cdot \vec{U} \equiv \frac{dU^\mu}{d\tau} U_\mu = \frac{1}{2} \frac{d}{d\tau} (U^\mu U_\mu) = \frac{1}{2} \frac{d}{d\tau} (-c^2) = 0$$

- We can also define the **four-force**  $F^\mu$  from the Lorentz force, so as to obtain a relativistic form of Newton's equation.

$$F^\mu \equiv m_0 a^\mu = \frac{dP^\mu}{d\tau}$$

$$\vec{F} = \frac{d\vec{P}}{d\tau} = \gamma \frac{d\vec{P}}{dt} = \gamma \left( \frac{1}{c} \frac{dE}{dt}, \frac{d\mathbf{p}}{dt} \right)$$

- Since  $\mathbf{F}_{\text{Lorentz}} = q \left[ \mathbf{E} + \frac{1}{c} (\mathbf{v} \times \mathbf{B}) \right]$ , the **Lorentz four-force** should involve (1) the electromagnetic field tensor and (2) the four-velocity and should also be (3) a four-vector and (4) proportional to the charge of the particle. Therefore, the simplest possibility is

$$F_{\text{Lorentz}}^\mu = \frac{q}{c} F^{\mu\nu} U_\nu$$

- Let's check to see if it is indeed what we want.

$$\frac{dW}{dt} = \mathbf{F} \cdot \mathbf{v} = q\mathbf{E} \cdot \mathbf{v}$$

$$F_{\text{Lorentz}}^0 = \frac{q}{c} F^{0\nu} U_\nu = \frac{q}{c} \sum_{i=1}^3 E_i \gamma v_i = \frac{q}{c} \gamma (\mathbf{E} \cdot \mathbf{v}) \longrightarrow \frac{dE}{dt} = q\mathbf{E} \cdot \mathbf{v} : \text{conservation of energy}$$

The rate of change of particle energy is the mechanical work done on the particle by the field.

$$\begin{aligned} F_{\text{Lorentz}}^1 &= \frac{q}{c} F^{1\nu} U_\nu = \frac{q}{c} (F^{10} (-\gamma c) + F^{12} \gamma v_2 + F^{13} \gamma v_3) \\ &= \frac{q}{c} \gamma (E_1 c + B_3 v_2 - B_2 v_3) \end{aligned} \longrightarrow \frac{d\mathbf{p}}{dt} = q \left( \mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right)$$

Therefore, we obtained the desired expression for the four-Lorentz force.

- Note that **the four-force is always orthogonal to the four-velocity**:

$$\vec{F} \cdot \vec{U} = m_0 (\vec{a} \cdot \vec{U}) = 0$$

It implies that **every four-force must have some velocity dependence**, although this dependence might become negligible in the nonrelativistic limit.

For the Lorentz four-force, in particular, we find

$$\vec{F}_{\text{Lorentz}} \cdot \vec{U} = \frac{q}{c} F^{\mu\nu} U_\mu U_\nu = 0$$

because  $F^{\mu\nu}$  is antisymmetric and  $U_\mu U_\nu$  is symmetric.



# Mathematical Formulae

- Gamma function

$$\Gamma(x) \equiv \int_0^{\infty} t^{x-1} e^{-t} dt, \quad \Gamma(x) = (x-1)! = (x-1)\Gamma(x-2), \quad \Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$$

- Euler-Mascheroni constant

$$\gamma \equiv \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n \frac{1}{k} - \ln(n) \right) = - \int_0^{\infty} e^{-x} \ln x dx = 0.577215664901532$$

- Modified Bessel function of the second kind

$$K_n(x) \equiv \frac{\Gamma(n+1/2)(2x)^n}{\sqrt{\pi}} \int_0^{\infty} \frac{\cos t}{(t^2 + x^2)^{n+1/2}} dt$$

$$(1) \quad 0 < x < \sqrt{n+1}$$

$$K_n(x) \approx \begin{cases} -\ln\left(\frac{x}{2}\right) - \gamma & \text{if } n = 0 \\ \frac{\Gamma(n)}{2} \left(\frac{2}{x}\right)^n & \text{if } n > 0 \end{cases}$$

$$(2) \quad x \gg |n^2 - 1/4|$$

$$K_n(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x} \left[ 1 + \frac{(4n^2 - 1)}{8x} \right]$$

## Recurrence formulae

$$K_{n-1}(x) - K_{n+1}(x) = -\frac{2n}{x} K_n(x)$$

$$K_{n-1}(x) + K_{n+1}(x) = -2K'_n(x)$$

## Integral formula

$$\begin{aligned} \int x K_n^2(x) dx &= \frac{1}{2} x^2 [K_n^2(x) - K_{n-1}(x) K_{n+1}(x)] \\ &= -x K_{n-1}(x) K_n(x) + \\ &\quad \frac{1}{2} x^2 [K_n^2(x) - K_{n-1}^2(x)] \end{aligned}$$

# [Fields of a Uniformly Moving Charge]

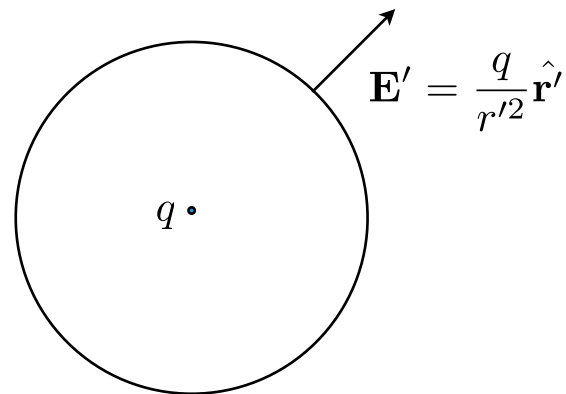
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- Recall that the electromagnetic field from a single moving charge is given by

$$\begin{aligned}
 &\text{velocity field} && \text{acceleration field} \\
 \mathbf{E}(\mathbf{r}, t) = &q \left[ \frac{(\mathbf{n} - \boldsymbol{\beta})(1 - \beta^2)}{\kappa^3 R^2} \right]_{\text{ret}} + \frac{q}{c} \left[ \frac{\mathbf{n}}{\kappa^3 R} \times \left\{ (\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right\} \right]_{\text{ret}} \\
 \mathbf{B}(\mathbf{r}, t) = &[\mathbf{n} \times \mathbf{E}(\mathbf{r}, t)]_{\text{ret}}
 \end{aligned}$$

- The velocity field is caused by a motion of the charge relative to the observer frame. Hence, the velocity field would be obtained by Lorentz transforming the static electromagnetic field (Coulomb field).

- Let's find the fields of a charge moving with constant velocity  $v$  along the  $x$  axis. In the rest frame  $K'$  of the particle the fields are given by



**Coulomb field in  $K'$**

$$\mathbf{E}' = (E'_x, E'_y, E'_z) = \frac{q}{r'^3} (x', y', z') \quad \text{where} \quad r' = (x'^2 + y'^2 + z'^2)^{1/2}$$

$$\mathbf{B}' = (0, 0, 0)$$

Inverse transformation of the previous one:

$$\begin{aligned} \mathbf{E}_{\parallel} &= \mathbf{E}'_{\parallel} & \mathbf{B}_{\parallel} &= \mathbf{B}'_{\parallel} \\ \mathbf{E}_{\perp} &= \gamma (\mathbf{E}'_{\perp} - \boldsymbol{\beta} \times \mathbf{B}') & \mathbf{B}_{\perp} &= \gamma (\mathbf{B}'_{\perp} + \boldsymbol{\beta} \times \mathbf{E}') \end{aligned}$$



$$\begin{aligned} E_x &= \frac{qx'}{r'^3} & B_x &= 0 \\ E_y &= \gamma \frac{qy'}{r'^3} & B_y &= -\gamma\beta \frac{qz'}{r'^3} \\ E_z &= \gamma \frac{qz'}{r'^3} & B_z &= \gamma\beta \frac{qy'}{r'^3} \end{aligned}$$

Now, we need to express these equations in terms of the unprimed coordinates. Since  $x' = \gamma(x - vt)$ ,  $y' = y$ ,  $z' = z$ , we obtain

$$\begin{aligned} E_x &= \gamma \frac{q(x - vt)}{r^3} & B_x &= 0 \\ E_y &= \gamma \frac{qy}{r^3} & B_y &= -\gamma\beta \frac{qz}{r^3} \\ E_z &= \gamma \frac{qz}{r^3} & B_z &= \gamma\beta \frac{qy}{r^3} \end{aligned}$$

$$\text{where} \quad r = \left[ \gamma^2 (x - vt)^2 + y^2 + z^2 \right]^{1/2} \quad (= r')$$

**Question:**

Is this equivalent to the velocity field given by the Lienard-Wiechert potentials?

# - Velocity field from the retarded potential

- For simplicity, assume  $z = 0$ .

$$\begin{aligned}\mathbf{E} &= (E_x, E_y, E_z) = \gamma \frac{q}{r^3} (x - vt, y, z) \\ &= \gamma \frac{q}{(\gamma^2 \bar{x}^2 + y^2)^{3/2}} (\bar{x}, y, 0) \quad \text{where } \bar{x} \equiv x - vt\end{aligned}$$

Let us first find where the **retarded position** of the particle is.

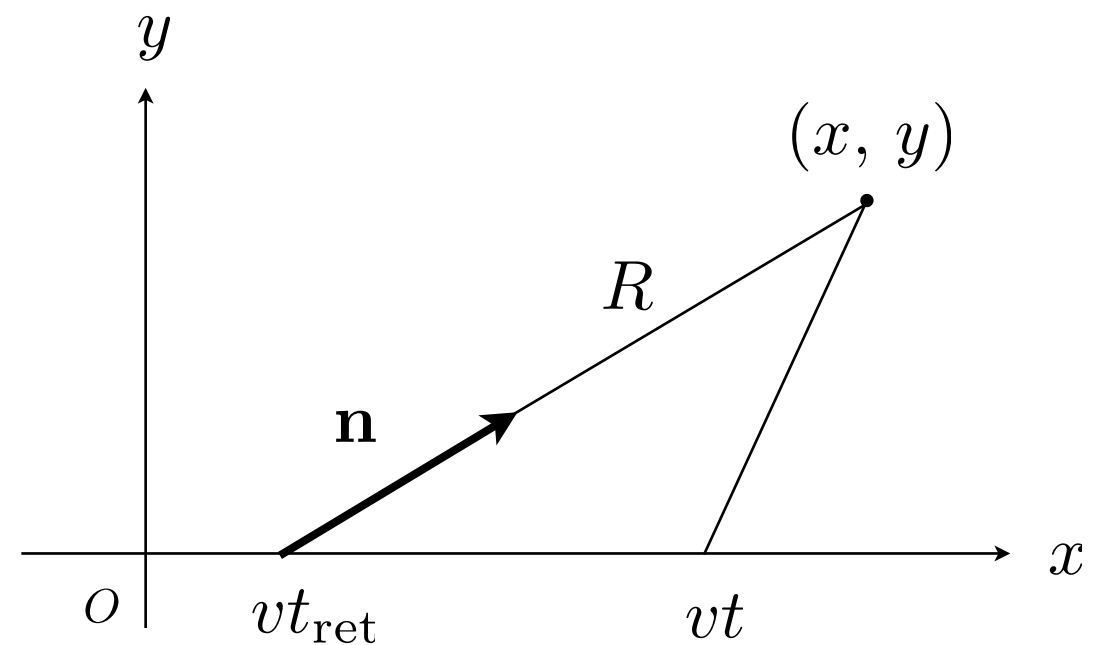
$$t_{\text{ret}} \equiv t - R/c$$

$$R^2 = (x - vt_{\text{ret}})^2 + y^2 = (\bar{x} + \beta R)^2 + y^2$$

$$\begin{aligned}R \longrightarrow (1 - \beta^2) R^2 - 2\bar{x}\beta R - \bar{x}^2 - y^2 &= 0 \\ R^2 - 2\bar{x}\gamma^2\beta R - \gamma^2(\bar{x}^2 + y^2) &= 0\end{aligned}$$

$$\begin{aligned}R &= \gamma^2\beta\bar{x} \pm [\gamma^4\beta^2\bar{x}^2 + \gamma^2(\bar{x}^2 + y^2)]^{1/2} \\ &= \gamma^2\beta\bar{x} \pm \gamma [\gamma^2\beta^2\bar{x}^2 + (\bar{x}^2 + y^2)]^{1/2} \\ &= \gamma^2\beta\bar{x} \pm \gamma (\gamma^2\bar{x}^2 + y^2)^{1/2}\end{aligned}$$

$$\text{positive solution} \rightarrow R = \gamma^2\beta\bar{x} + \gamma (\gamma^2\bar{x}^2 + y^2)^{1/2}$$



- Let us first find where the **retarded position** of the particle is.

$$(1) \quad \mathbf{n} = \frac{(x - vt + vR/c) \hat{\mathbf{x}} + y \hat{\mathbf{y}}}{R}$$

$$= \frac{(\bar{x} + \beta R)}{R} \hat{\mathbf{x}} + \frac{y}{R} \hat{\mathbf{y}}$$

$$= \left( \frac{\bar{x}}{R} + \beta \right) \hat{\mathbf{x}} + \frac{y}{R} \hat{\mathbf{y}}$$

$$\mathbf{n} - \boldsymbol{\beta} = \frac{\bar{x}}{R} \hat{\mathbf{x}} + \frac{y}{R} \hat{\mathbf{y}}$$

$$\boldsymbol{\beta} = \beta \hat{\mathbf{x}}$$

$$\mathbf{E} = \gamma \frac{qR}{(\gamma^2 \bar{x}^2 + y^2)^{3/2}} (\mathbf{n} - \boldsymbol{\beta})$$

$$\mathbf{n} \cdot \boldsymbol{\beta} = \beta^2 + \frac{\bar{x}}{R} \beta$$

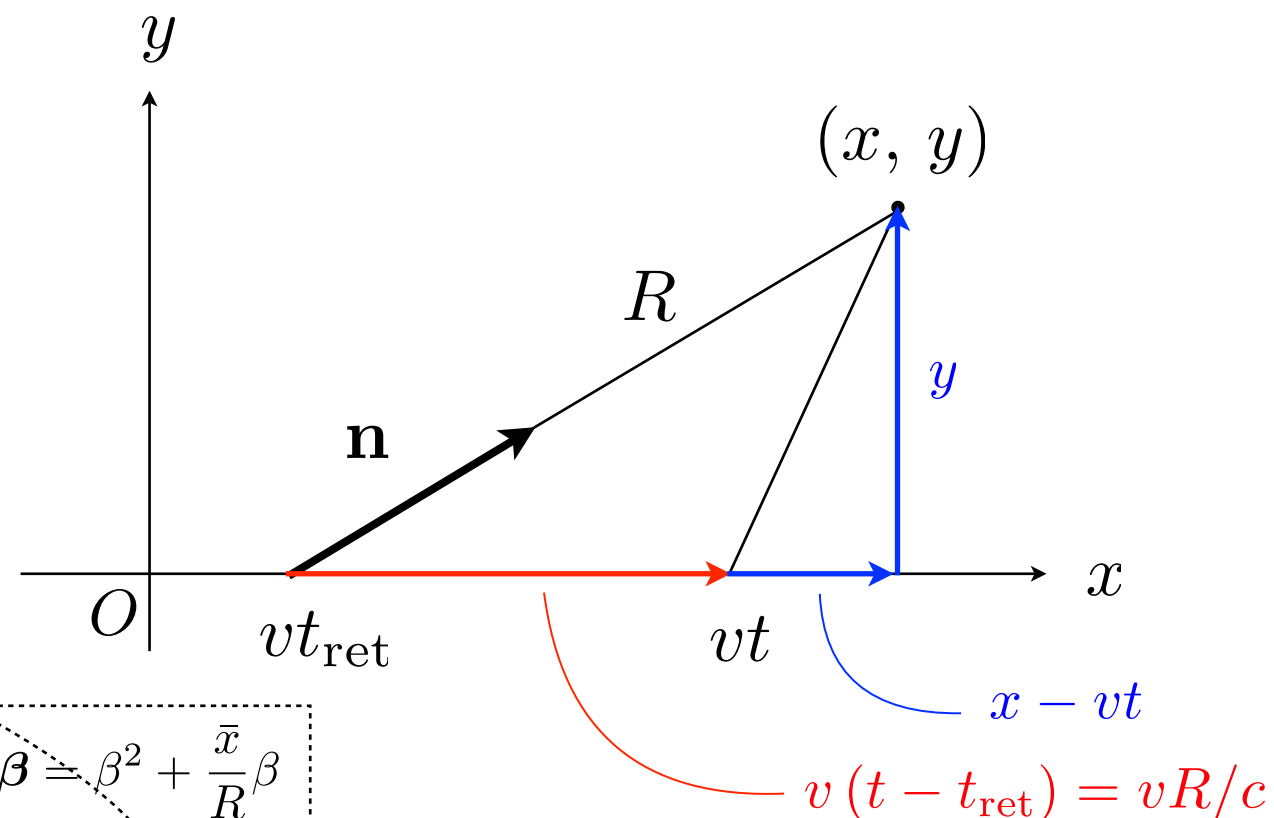
$$(2) \quad (\gamma^2 \bar{x}^2 + y^2)^{1/2} = \frac{R - \gamma^2 \beta \bar{x}}{\gamma} = R\gamma \left( \frac{1}{\gamma^2} - \frac{\beta \bar{x}}{R} \right)$$

$$= R\gamma \left( 1 - \beta^2 - \frac{\beta \bar{x}}{R} \right)$$

$$= R\gamma \left[ 1 - \beta \left( \frac{\bar{x}}{R} + \beta \right) \right]$$

$$= R\gamma (1 - \mathbf{n} \cdot \boldsymbol{\beta}) = R\gamma \kappa$$

$$R = \gamma^2 \beta \bar{x} + \gamma (\gamma^2 \bar{x}^2 + y^2)^{1/2}$$



$$\therefore \mathbf{E} = q \frac{(\mathbf{n} - \boldsymbol{\beta})}{\gamma^2 \kappa^3 R^2} = q \frac{(\mathbf{n} - \boldsymbol{\beta}) (1 - \beta^2)}{\kappa^3 R^2}$$

This is identical to the velocity field component!

# - Time-dependence of the electric field at a point

- For simplicity, let us choose the field point to be at  $(0, b, 0)$ . This involves no loss in generality. Then,

$$E_x = -\frac{q\gamma vt}{(\gamma^2 v^2 t^2 + b^2)^{3/2}} = -\frac{q}{b^2} \frac{\gamma vt/b}{(\gamma^2 v^2 t^2/b^2 + 1)^{3/2}}$$

$$E_y = \frac{q\gamma b}{(\gamma^2 v^2 t^2 + b^2)^{3/2}} = \frac{q\gamma}{b^2} \frac{1}{(\gamma^2 v^2 t^2/b^2 + 1)^{3/2}}$$

$$E_z = 0$$

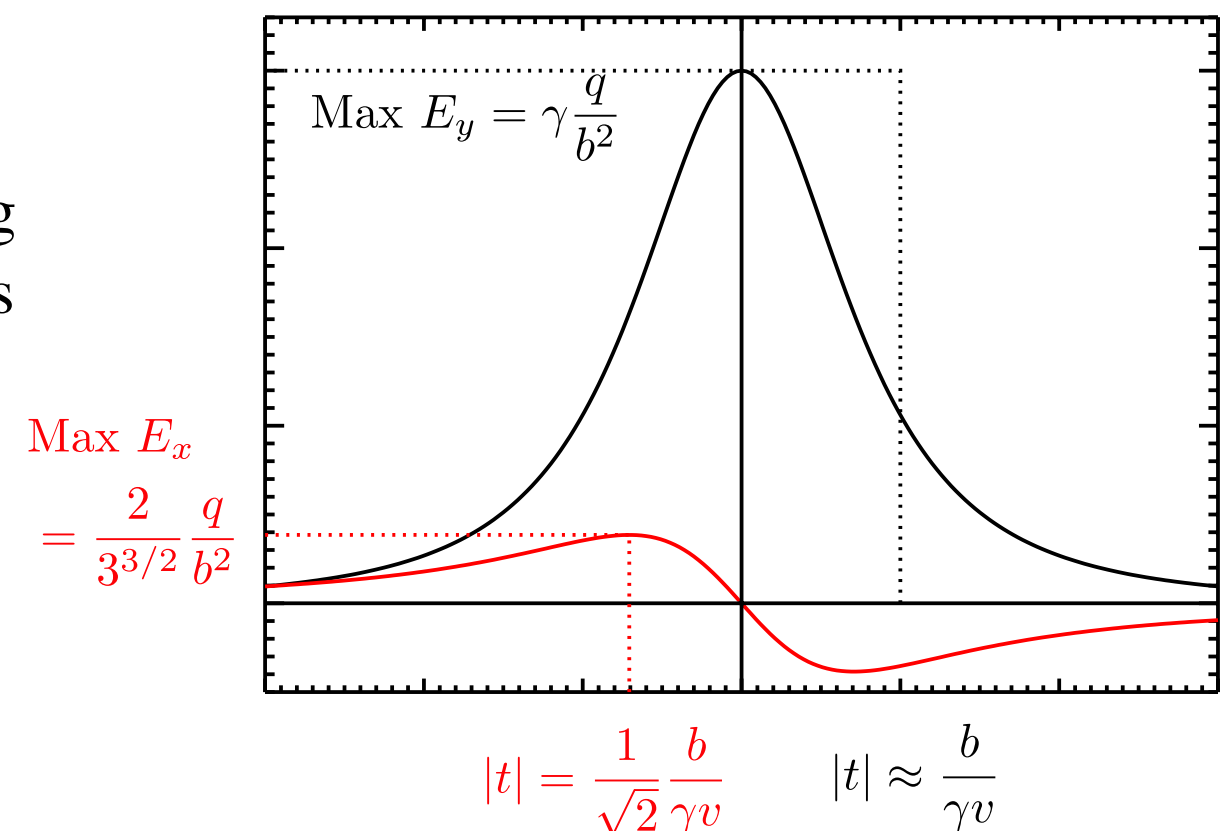
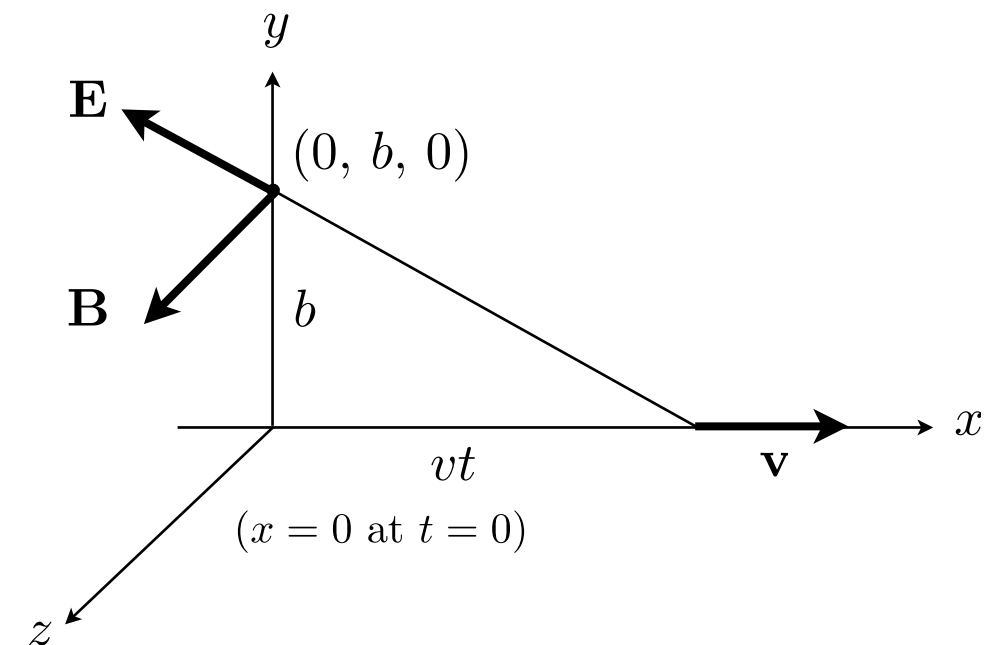
$$B_x = 0, \quad B_y = 0, \quad B_z = \beta E_y$$

- (1) The fields are strong only during a time interval of  $|vt| \lesssim b/\gamma$ , implying that the field of the moving charge are concentrated in the plane transverse to its motion into an angle of order  $1/\gamma$ .

$$\Delta\theta \approx v\Delta t/b \sim 1/\gamma$$

- (2) As  $\gamma \gg 1 \rightarrow |E_x|_{\max} \ll |E_y|_{\max}$

The field of a highly relativistic charge appears to be a pulse of radiation traveling in the same direction as the charge and confined to the transverse plane.



see the animation and python code:

[https://seoncafe.github.io/Teaching\\_files/2023b\\_astrophysics/ani\\_pulse.mp4](https://seoncafe.github.io/Teaching_files/2023b_astrophysics/ani_pulse.mp4)

[https://seoncafe.github.io/Teaching\\_files/2023b\\_astrophysics/plot\\_pulse.py](https://seoncafe.github.io/Teaching_files/2023b_astrophysics/plot_pulse.py)

# - Spectrum of the pulse

- Spectrum of this pulse is given by (if the  $x$ -axis component is ignored)

$$\begin{aligned}
 \hat{E}(\omega) &= \frac{1}{2\pi} \int E_y(t) e^{i\omega t} dt \\
 &= \frac{q\gamma b}{2\pi} \int_{-\infty}^{\infty} (\gamma^2 v^2 t^2 + b^2)^{-3/2} e^{i\omega t} dt = \frac{q\gamma b}{2\pi} \int_0^{\infty} (\gamma^2 v^2 t^2 + b^2)^{-3/2} (e^{i\omega t} + e^{-i\omega t}) dt \\
 &= \frac{q\gamma b}{\pi} \int_0^{\infty} (\gamma^2 v^2 t^2 + b^2)^{-3/2} \cos \omega t dt
 \end{aligned}$$

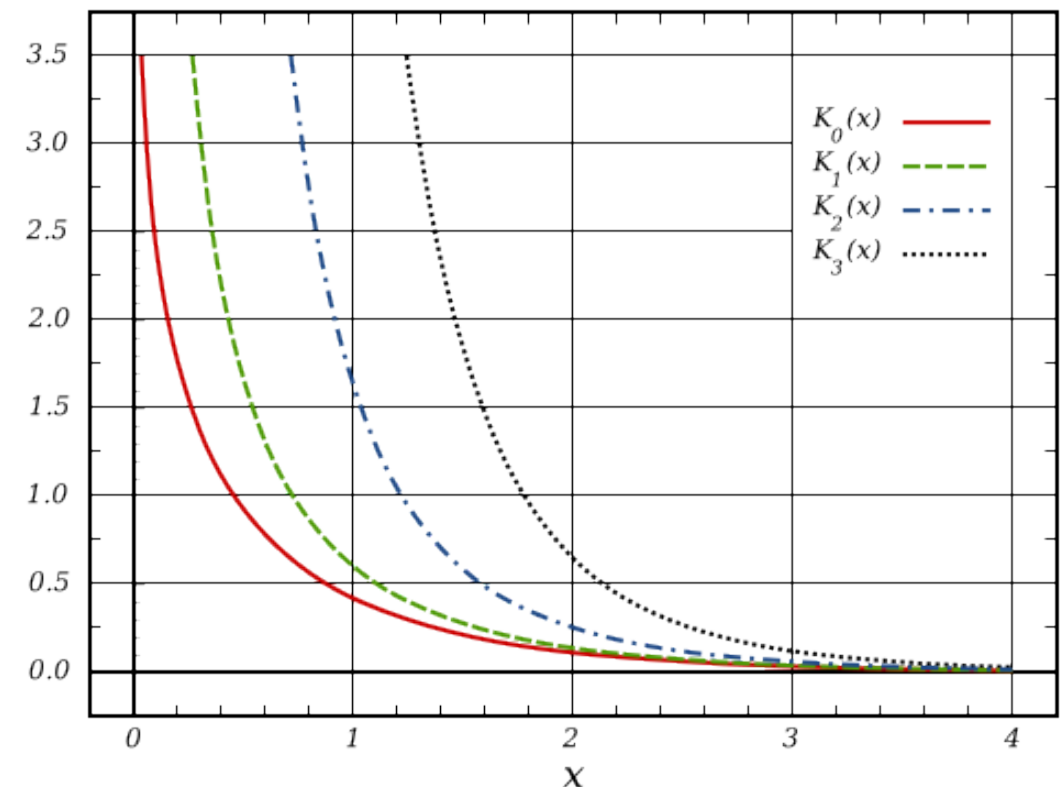
This integral can be done in terms of the modified Bessel function:

$$K_n(x) \equiv \frac{\Gamma(n + 1/2)(2x)^n}{\sqrt{\pi}} \int_0^{\infty} \frac{\cos t}{(t^2 + x^2)^{n+1/2}} dt \quad \text{Gamma function: } \Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$$

$$\begin{aligned}
 \hat{E}(\omega) &= \frac{q\gamma b}{\pi} \left( \frac{\gamma^2 v^2}{\omega^2} \right)^{-3/2} \frac{1}{\omega} \int_0^{\infty} \left( \omega^2 t^2 + \frac{b^2 \omega^2}{\gamma^2 v^2} \right)^{-3/2} (\cos \omega t) d(\omega t) \\
 &= \frac{q}{\pi b v} \frac{b \omega}{\gamma v} K_1 \left( \frac{b \omega}{\gamma v} \right)
 \end{aligned}$$

Thus the spectrum is

$$\frac{dW}{dA d\omega} = c \left| \hat{E}(\omega) \right|^2 = \frac{q^2}{\pi^2 b^2 v^2} \left( \frac{b \omega}{\gamma v} \right)^2 K_1^2 \left( \frac{b \omega}{\gamma v} \right)$$



- The spectrum starts cut off for  $\omega > \gamma v/b$ .

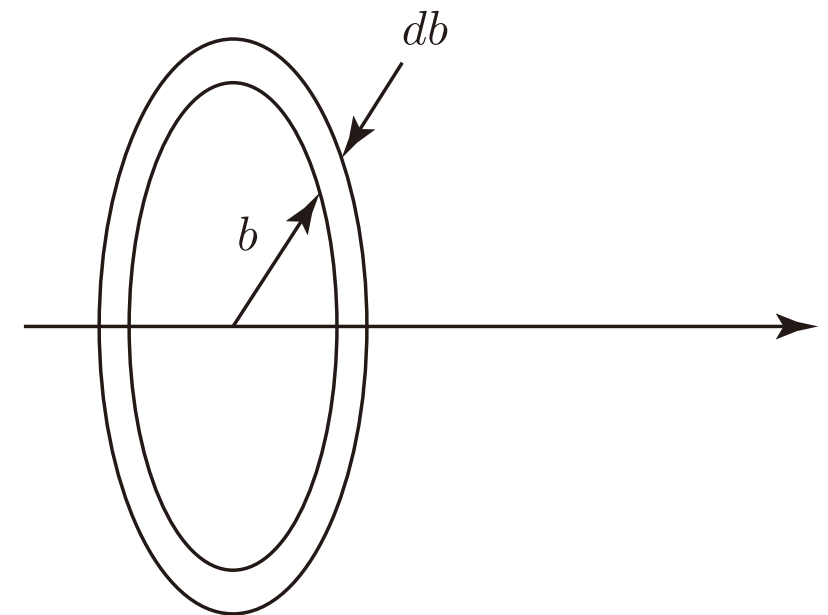
This is predicted from the uncertainty principle, since the pulse is confined roughly to a time interval of order  $\Delta t \approx b/\gamma v$ .

$$\Delta\omega \sim \frac{1}{\Delta t} \sim \gamma v/b$$

- Total energy per unit frequency range is obtained by

$$\frac{dW}{d\omega} = 2\pi \int_{b_{\min}}^{b_{\max}} \frac{dW}{dA d\omega} b db$$

$$b_{\max} \rightarrow \infty, \quad b_{\min} \neq 0 \text{ (a finite value)}$$



The lower limit has been chosen not as zero but as some minimum distance  $b_{\min}$ , such that the approximation of the field by means of classical electrodynamics and a point charge is valid.

$$\begin{aligned} \frac{dW}{d\omega} &= \frac{2q^2 c}{\pi v^2} \int_x^\infty y K_1^2(y) dy \\ &= \frac{2q^2 c}{\pi v^2} \left[ x K_0(x) K_1(x) - \frac{1}{2} x^2 (K_1^2(x) - K_0^2(x)) \right] \end{aligned} \quad \text{where} \quad y \equiv \frac{\omega b}{\gamma v} \quad \text{and} \quad x \equiv \frac{\omega b_{\min}}{\gamma v}$$



- Two limiting cases:

$$(1) \quad \omega \ll \frac{\gamma v}{b_{\min}} \quad (x \ll 1),$$

$$\begin{aligned} & xK_0(x)K_1(x) - \frac{1}{2}x^2 (K_1^2(x) - K_0^2(x)) \\ & \approx x(-\ln(x/2) - \gamma) \frac{1}{x} - \frac{x^2}{2} \left[ \frac{1}{x^2} - (\ln(x/2) + \gamma)^2 \right] \\ & \approx \ln \left[ \frac{2}{x} e^{-(\gamma+1/2)} \right] \\ & = \ln \left( \frac{0.68}{x} \right) \end{aligned} \quad \longrightarrow$$

$$\begin{aligned} & \omega \ll \frac{\gamma v}{b_{\min}} \\ & \frac{dW}{d\omega} = \frac{2q^2 c}{\pi v^2} \ln \left( 0.68 \frac{\gamma v}{\omega b_{\min}} \right) \end{aligned}$$

$$(2) \quad \omega \gg \frac{\gamma v}{b_{\min}} \quad (x \gg 1),$$

$$\begin{aligned} & xK_0(x)K_1(x) - \frac{1}{2}x^2 (K_1^2(x) - K_0^2(x)) \\ & \approx x \frac{\pi}{2x} e^{-2x} - \frac{1}{2}x^2 \frac{\pi}{2x} e^{-2x} \left[ \left( \frac{3}{8x} \right)^2 - \left( \frac{1}{8x} \right)^2 \right] \\ & = \frac{\pi}{4} e^{-2x} \end{aligned} \quad \longrightarrow$$

$$\begin{aligned} & \omega \gg \frac{\gamma v}{b_{\min}} \\ & \frac{dW}{d\omega} = \frac{q^2 c}{2v^2} \exp \left( -\frac{2\omega b_{\min}}{\gamma v} \right) \end{aligned}$$

# [Emission from Relativistic Particles]

- Total emitted power:

Imagine **an instantaneous rest frame**  $K'$ , such that the particle has zero velocity at a certain time. We can then calculate the radiation emitted by use of the dipole (Larmor) formula.

Suppose that the particle emits a total amount of energy  $dW'$  in this frame in time  $dt'$ . The momentum of this radiation is zero,  $d\mathbf{p}' = 0$ , **because the emission is symmetrical in the frame.**

The energy in a frame  $K$  moving with velocity  $-\mathbf{v}$  w.r.t. the particle is:

$$dW = \gamma dW' \longleftarrow dE = cdP^0 = c\tilde{\Lambda}^0_{\mu} dP'^{\mu} = c\tilde{\Lambda}^0_0 dP^0 = \gamma dE'$$

The time interval  $dt$  is simply

$$dt = \gamma dt'$$

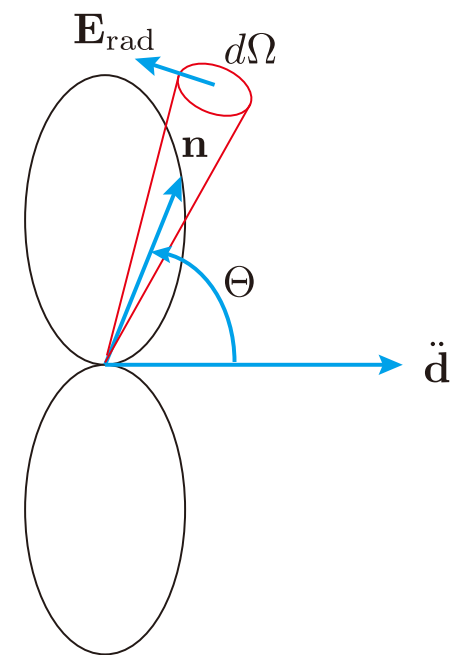
The total power emitted in frames  $K$  and  $K'$  are given by

$$P = \frac{dW}{dt}, \quad P' = \frac{dW'}{dt'}$$

Thus **the total emitted power is a Lorentz invariant** for any emitter that emits with front-back symmetry in its instantaneous rest frame.

$$P = P'$$

Note that if the radiation is asymmetric, then  $d\mathbf{p}' \neq 0$  and  $dW \neq \gamma dW'$ .



- **The Larmor formula in covariant form:**

Recall that  $\vec{a} \cdot \vec{U} = 0$ , and because  $\vec{U} = (c, \mathbf{0})$  in the instantaneous rest frame of the particle, we have

$$a'_0 = 0 \quad \rightarrow \quad |\mathbf{a}'|^2 = a'_k a'^k = a'_\mu a'^\mu = \vec{a} \cdot \vec{a}$$

Therefore,

$$\underset{\text{nonrelativistic}}{P' = \frac{2q^2}{3c^3} |\mathbf{a}'|^2} \quad \longrightarrow \quad \underset{\text{relativistic}}{P = \frac{2q^2}{3c^3} \vec{a} \cdot \vec{a}}$$

- Expression of  $P$  in terms of the three-vector acceleration

Recall

$$\begin{array}{l} \boxed{\begin{array}{l} dt = \gamma \left( dt' + \frac{v}{c^2} dx'_{\parallel} \right) \\ u_{\parallel} = \frac{u'_{\parallel} + v}{1 + vu'_{\parallel}/c^2} \\ u_{\perp} = \frac{u'_{\perp}}{\gamma \left( 1 + vu'_{\parallel}/c^2 \right)} \end{array}} \quad \longrightarrow \quad \begin{array}{l} \sigma \equiv \left( 1 + vu'_{\parallel}/c^2 \right) \\ dt = \gamma dt' \sigma \\ u_{\parallel} = \frac{u'_{\parallel} + v}{\sigma} \\ u_{\perp} = \frac{u'_{\perp}}{\gamma \sigma} \end{array} \quad \longrightarrow \quad \begin{array}{l} dt = \gamma dt' \sigma \\ du_{\parallel} = \frac{du'_{\parallel}}{\sigma} - \frac{u'_{\parallel} + v}{\sigma^2} \frac{v}{c^2} du'_{\parallel} \\ \quad = \frac{du'_{\parallel}}{\sigma^2} \left( 1 - \frac{v^2}{c^2} \right) = \frac{du'_{\parallel}}{\gamma^2 \sigma^2} \\ du_{\perp} = \frac{du'_{\perp}}{\gamma \sigma} - \frac{u'_{\perp}}{\gamma \sigma^2} \frac{v}{c^2} du'_{\parallel} \\ \quad = \frac{1}{\gamma \sigma^2} \left( \sigma du'_{\perp} - \frac{vu'_{\perp}}{c^2} du'_{\parallel} \right) \end{array}$$

Hence,

Transformation of three-vector acceleration:

$$a_{\parallel} = \frac{du_{\parallel}}{dt} = \frac{1}{\gamma^3 \sigma^3} \frac{du'_{\parallel}}{dt'}$$

$$a_{\perp} = \frac{du_{\perp}}{dt} = \frac{1}{\gamma^2 \sigma^3} \left( \sigma \frac{du'_{\perp}}{dt'} - \frac{vu'_{\perp}}{c^2} \frac{du'_{\parallel}}{dt'} \right)$$



$$a_{\parallel} = \frac{1}{\gamma^3 \sigma^3} a'_{\parallel}$$

$$a_{\perp} = \frac{1}{\gamma^2 \sigma^3} \left( \sigma a'_{\perp} - \frac{vu'_{\perp}}{c^2} a'_{\parallel} \right)$$

where  $\sigma \equiv \left( 1 + \frac{vu'_{\parallel}}{c^2} \right)$

In an instantaneous rest frame of a particle,

$$u'_{\parallel} = u'_{\perp} = 0, \quad \sigma = 1$$

$$a'_{\parallel} = \gamma^3 a_{\parallel}$$

$$a'_{\perp} = \gamma^2 a_{\perp}$$

Note  $\tan \theta'_a \equiv \frac{a'_{\perp}}{a'_{\parallel}} = \frac{1}{\gamma} \frac{a_{\perp}}{a_{\parallel}} = \frac{1}{\gamma} \tan \theta_a$

Thus we can write

$$P = \frac{2q^2}{3c^3} |\mathbf{a}'|^2 = \frac{2q^2}{3c^3} (a'^2_{\parallel} + a'^2_{\perp})$$

nonrelativistic

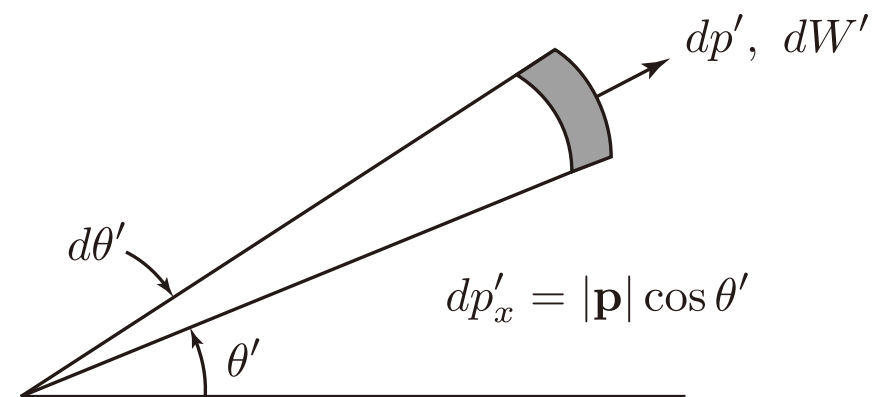
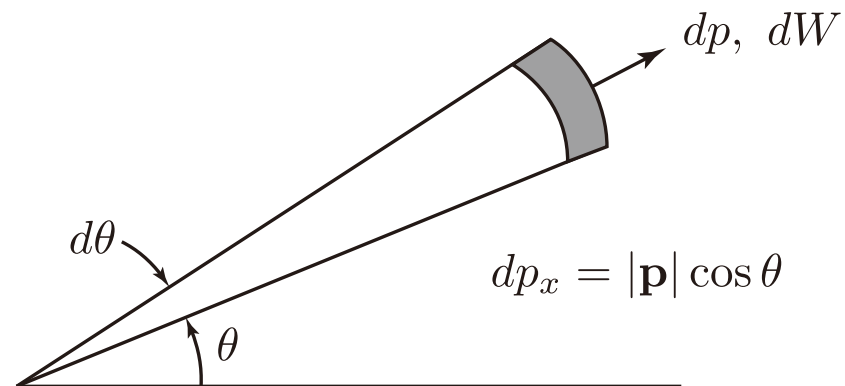


$$P = \frac{2q^2}{3c^3} \gamma^4 \left( \gamma^2 a^2_{\parallel} + a^2_{\perp} \right)$$

relativistic

# Differential Power

- Angular Distribution of Emitted and Received Power**



Note:  $d\phi' = d\phi$

In the instantaneous rest frame of the particle, let us consider an amount of energy  $dW'$  that is emitted into the solid angle  $d\Omega' = \sin \theta' d\theta' d\phi'$  (see the above figure).

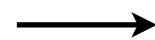
$$\begin{aligned}\mu &\equiv \cos \theta \quad \rightarrow \quad d\Omega = d\mu d\phi \\ \mu' &\equiv \cos \theta' \quad \rightarrow \quad d\Omega' = d\mu' d\phi'\end{aligned}$$

Recall the aberration formula:

$$\cos \theta = \frac{\cos \theta' + \beta}{1 + \beta \cos \theta'} \quad \rightarrow \quad \mu = \frac{\mu' + \beta}{1 + \beta \mu'} \quad \rightarrow \quad \mu' = \frac{\mu - \beta}{1 - \beta \mu}$$

$$d\mu = \frac{d\mu'}{\gamma^2 (1 + \beta \mu')^2}$$

$$d\mu' = \frac{d\mu}{\gamma^2 (1 - \beta \mu)^2}$$



$$d\Omega = \frac{d\Omega'}{\gamma^2 (1 + \beta \mu')^2}$$

$$d\Omega' = \frac{d\Omega}{\gamma^2 (1 - \beta \mu)^2}$$

Recall that energy and momentum form a four-vector

$$\vec{P} = (E/c, \mathbf{p}) \quad \text{and} \quad |\mathbf{p}| = E/c \quad \longrightarrow \quad dW = \gamma(dW' + v dp'_x) = \gamma(1 + \beta\mu')dW'$$

$$\therefore dW = \gamma(1 + \beta\mu')dW', \quad dW' = \gamma(1 - \beta\mu)dW$$

$$\frac{dW}{d\Omega} = \gamma^3 (1 + \beta\mu')^3 \frac{dW'}{d\Omega'}, \quad \frac{dW'}{d\Omega'} = \gamma^3 (1 - \beta\mu)^3 \frac{dW}{d\Omega}$$

In the rest frame, **the power emitted in a unit time interval** is  $\frac{dP'}{d\Omega'} \equiv \frac{dW'}{dt' d\Omega'}$

However, in the observer's frame, there are two possible choices for the time interval to calculate the power.

(1)  $dt = \gamma dt'$ :

This is the time interval during which the emission occurs. With this choice we obtain **the emitted power**.

(2)  $dt_A = \gamma(1 - \beta\mu)dt'$  or  $dt_A = \gamma^{-1}(1 + \beta\mu')^{-1}dt'$ :

This is the time interval of the radiation as received by a stationary observer in  $K$ . With this choice we obtain **the received power**.

- Thus we obtain the two results:

$$\begin{aligned}\frac{dP_e}{d\Omega} &= \gamma^2 (1 + \beta\mu')^3 \frac{dP'}{d\Omega'} = \frac{1}{\gamma^4 (1 - \beta\mu)^3} \frac{dP'}{d\Omega'} \\ \frac{dP_r}{d\Omega} &= \gamma^4 (1 + \beta\mu')^4 \frac{dP'}{d\Omega'} = \frac{1}{\gamma^4 (1 - \beta\mu)^4} \frac{dP'}{d\Omega'}\end{aligned}$$

$P_r$  is the power actually measured by an observer. It has the expected symmetry property of yielding the inverse transformation by interchanging primed and unprimed variables, along with a change of sign of  $\beta$ .

$P_e$  is used in the discussion of emission coefficient.

**In practice, the distinction between emitted and received power is often not important, since they are equal in an average sense for stationary distributions of particles.**

- Beaming effect:

If the radiation is isotropic in the particle's frame, then the angular distribution in the observer's frame will be highly peaked in the forward direction for highly relativistic velocities.

The factor  $\gamma^{-4} (1 - \beta\mu)^{-4}$  is sharply peaked near  $\theta \approx 0$  with an angular scale of order  $1/\gamma$ .

$$\gamma^{-4} (1 - \beta\mu)^{-4} \approx \gamma^{-4} \left[ 1 - \left( 1 - \frac{1}{2\gamma^2} \right) \left( 1 - \frac{\theta^2}{2} \right) \right]^{-4} = \gamma^{-4} \left( \frac{1}{2\gamma^2} + \frac{\theta^2}{2} \right)^{-4} = \left( \frac{2\gamma}{1 + \gamma^2\theta^2} \right)^4$$

# Homework (due date: 10/31)

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[Q8] Problem 4.13 (page 153) in Radiative Processes in Astrophysics (Rybicki & Lightman)

- (1) Show that an observer moving with respect to a blackbody field of temperature  $T$  will see blackbody radiation with a temperature that depends on angle according to

$$T' = \frac{(1 - v^2/c^2)^{1/2}}{1 + (v/c) \cos \theta'} T.$$

- (2) The isotropy of the 2.7 K universal blackbody radiation at  $\lambda = 3$  cm has been established to about one part in  $10^3$ . What is the maximum velocity that the earth can have with respect to the frame in which this radiation isotropic? Hint: Isotropy is measured by the following ratio:

$$P_{\text{iso}} = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}}$$