$(AGN)^2$

4. Calculation of Emitted Spectrum

Week 5 & 6

April 8 (Monday), 2024

updated on 04/10, 14:54

선광일 (Kwangil Seon)

KASI / UST

Selection Rules

- **Selection Rules**

(1) one electron jumps
(2) Δn any
(3) $\Delta l = \pm 1$
(4) parity change

→ selection rules for configuration (rules 1–4)

(5) $\Delta S = 0$ → ***intercombination*** line if only this rule is violated.

(6) $\Delta L = 0, \ \pm 1$ (except $L = 0 - 0$)
(7) $\Delta J = 0, \ \pm 1$ (except $J = 0 - 0$)
(8) $\Delta F = 0, \ \pm 1$ (except $F = 0 - 0$) → This is not commonly considered.

- ***Allowed = Electric Dipole*** : Transitions which satisfy all the above selection rules are referred to as ***allowed transitions***. These transitions are strong and have a typical lifetime of $\sim 10^{-8}$ s. Allowed transitions are denoted without square brackets.

$$\text{e.g., C IV } 1548, 1550 \text{ Å}$$

- Photons do not change spin, so transitions usually occur between terms with the same spin state ($\Delta S = 0$). However, relativistic effects mix spin states, particularly for high Z atoms and ions. As a result, one can get (weak) spin changing transitions. These are called ***intercombination (semi-forbidden or intersystem) transitions*** or lines. They have a typical lifetime of $\sim 10^{-3}$ s. An intercombination transition is denoted with a single right bracket.

$$\text{C III] } 2s^2\ {}^1S - 2s2p\ {}^3P^o \text{ at } 1908.7\,\text{Å}. \quad (\Delta S = 1)$$

- If any one of the rules 1-4, 6-8 are violated, they are called ***forbidden transitions*** or lines. They have a typical lifetime of $\sim 1 - 10^3$ s. A forbidden transition is denoted with two square brackets.

$$1906.7\,\text{Å [C III] } 2s^2\ {}^1S_0 - 2s2p\ {}^3P^o_2, \quad (\Delta S = 1,\ \Delta J = 2)$$

- ***Resonance line*** denotes the longest wavelength, dipole-allowed transition arising from the ground state of a particular atom or ion.

4.1 Introduction

- Collisionally excited lines: Chief emission lines of gaseous nebulae
  - The bulk of the lines are collisionally excited lines, which arise from levels within a few eV of the gound level., and which can be excited by collisions with thermal electrons.
  - In the optical region, all these lines are forbidden lines, because in these ions the excited levels within a few eV of the ground level arise from the same electron configuration as the ground level itself. The radiative transitions are forbidden by the parity selection rule.
  - However, in the UV, collisioinally excited lines begin to appear as being permitted.

- Recombination lines of H I, He I, and He II
  - The are emitted by atoms undergoing radiative transitions in cascading down to the ground level following recombinations to excited levels.
  - These lines are characteristic features of the spectra of gaseouns nebulae.

- Continuum emission processes
  - bound-free emission (free-bound would be the better word.)
  - free-free emission

4.2 Optical Recombination Lines

- Populations of levels in LTE
  - In thermodynamics equilibrium (TE), the Saha equation gives the degree of ionization:

$$\frac{n_p n_e}{n_{1S}} = \left(\frac{2\pi m k T}{h^2}\right)^{3/2} \exp\left(-h\nu_0/kT\right) \quad (h\nu_0 = 1 \text{ Ryd})$$

https://casper.astro.berkeley.edu/astrobaki/index.php/Milne_Relation
https://casper.astro.berkeley.edu/astrobaki/index.php/Saha_Equation

and the Boltzmann equation gives the relative populations between levels:

$$\frac{n_{nL}}{n_{1S}} = (2L+1)\exp\left(-\chi_n/kT\right), \quad \left(\chi_n = \left|E_n - E_1\right| = \text{energy difference}\right)$$

$$\frac{n_j}{n_2} = \frac{g_j}{g_i}\exp\left(-E_{ji}/kT\right)$$

where the ratio of statistical weights is $g_{nL}/g_{1S} = (2L+1)$.

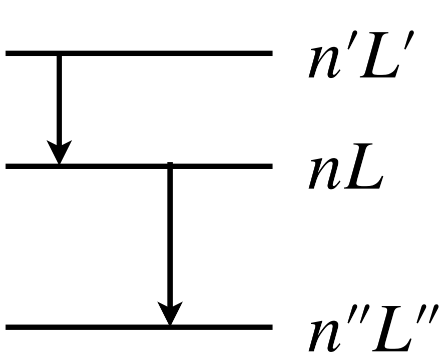
  - Combining these equation,

$$n_{nL} = (2L+1)\left(\frac{h^2}{2\pi m k T}\right)^{3/2} \exp\left(-X_n/kT\right) n_p n_e, \quad \text{where } X_n = h\nu_0 - \chi_n = \frac{h\nu_0}{n^2}.$$

the ionization potential of the level nL

- Statistical equilibrium for the population of any level nL
  - ***Case A: In the limit of very low density, the only processes that need be considered are captures and downward-radiative transitions.***
  - The equation of statistical equilibrium for an level nL is

$$n_p n_e \alpha_{nL}(T) + \sum_{n'>n}^{\infty} \sum_{L'} n_{n'L'} A_{n'L',nL} = n_{nL} \sum_{n''=1}^{n-1} \sum_{L''} A_{nL,n''L''}$$



In general, $A_{n'L',n''L'} \neq 0$ only if $L' = L'' + 1$.

selection rules: $\Delta L = 0, \pm 1$ & $\Delta l = \pm 1$ and $L = l$ for H

- Method (1): Non-LTE Departure Coefficients b_{nL}
  - Non-LTE departure coefficients b_{nL} = the dimensional factors that measure the deviation from thermodynamic equilibrium.
  - In general, in a non-LTE state, the population may be written

$$\boxed{n_{nL} = b_{nL}(2L+1)\left(\frac{h^2}{2\pi mkT}\right)^{3/2} \exp\left(-X_n/kT\right) n_p n_e,} \quad \text{and } b_{nL} = 1 \text{ in TE.}$$

  - Substituting this equation to the statistical balance equation, we obtain

$$\alpha_{nL}\frac{1}{(2L+1)}\left(\frac{2\pi mkT}{h^2}\right)^{3/2} e^{\left(-X_n/kT\right)} + \sum_{n'>n}^{\infty}\sum_{L''} b_{n'L'}A_{n'L',nL}\left(\frac{2L'+1}{2L+1}\right) e^{\left(X_{n'}-X_n\right)/kT}$$

$$= b_{nL}\sum_{n''=1}^{n-1}\sum_{L''} A_{nL,n''L''}$$

  - The b_{nL} factors are independent of density as long as recombination and downward-radiative transitions are the only relevant processes.
  - If b_{nL} are known for all $n \geq n_K$ and $L = 0, 1, \cdots, n-1$, then above equation can be solved for $n \leq n_K - 1$.

- Method (2): Probability matrix and Cascade matrix
  - **Probability matrix**, $P(nL, n'L')$ is the probability that population of nL is followed by a direct radiative transition to $n'L'$.

$$P_{nL,n'L'} = \frac{A_{nL,n'L'}}{\sum_{n''=1}^{n-1} \sum_{L''} A_{nL,n''L''}}, \quad \text{which is zero unless } L' = L \pm 1.$$

  - **Cascade matrix**, $C(nL, n'L')$ is the probability that population of nL is followed by a transition to $n'L'$ via all possible cascade routes.

$$C_{nL,n'L'} = \sum_{n''>n'}^{n} \sum_{L''=L'\pm 1} C_{nL,n''L''} P_{n''L'',n'L'} \quad \text{if we define} \quad C_{nL,nL''} = \delta_{LL''}$$

  - The solutions of the equilibrium equations my be written down as follows:

$$n_p n_e \sum_{n'=n}^{\infty} \sum_{L'=0}^{n'-1} \alpha_{n'L'}(T) C_{n'L',nL} = n_{nL} \sum_{n''=1}^{n-1} \sum_{L''=L\pm 1} A_{nL,n''L''}$$

  - Once the cascade matrix has been calculated, it can be used to find the b_{nL} factors or the populations n_{nL} at any temperature.
  - This is true even for cases in which the population occurs by other non-radiative processes (i.e., collisional excitation)

- Emission coefficient (emissivity)

$$j_{nn'} = \frac{h\nu_{nn'}}{4\pi} \sum_{L=0}^{n-1} \sum_{L'=L\pm1} n_{nL} A_{nL,n'L'}$$

  - The above situation is called Case A, which assumes that all line photons emitted in the nebula escape without absorption and therefore without causing further upward transitions.
  - Such nebulae can contain only a relatively small amount of gas and are mostly too faint to be easily observed.

- Central line-absorption cross section of Lyman resonance lines
  - Nebulae that contain observable amounts of gas generally have quite large optical depths in the Lyman resonance lines of H I.
  - The central line-absorption cross section for a Lyman resonance line (between n and 1) is

$$\sigma_0(Ln) = f_{nP,1S} \frac{\pi e^2}{m_e c} \frac{1}{\nu_{n1}(\upsilon_{\rm th}/c)} \frac{1}{\sqrt{\pi}} = \frac{3\lambda_{n1}^3}{8\pi} \left(\frac{m_{\rm H}}{2\pi kT} \right)^{1/2} A_{nP,1S} \quad [{\rm cm}^2]$$

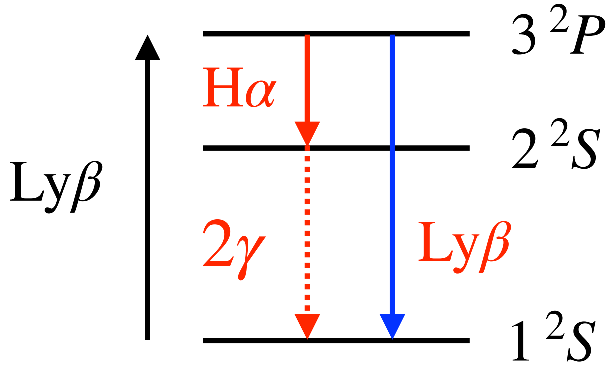
Here, $\upsilon_{\rm th} = \left(\frac{2kT}{m_{\rm H}} \right)^{1/2}$ is the thermal velocity and λ_{n1} is the wavelength of the line.

Relation between the absorption cross section and A-coefficient for a transition between 1 and 2 states:

$$f_{12} \frac{\pi e^2}{m_e c} = \frac{h\nu_{21}}{4\pi} B_{12} = \frac{h\nu_{21}}{4\pi} \frac{g_2}{g_1} \frac{c^2}{2h\nu_{21}^3} A_{21}\,, \qquad g_{nP} : g_{1S} = 3 : 1.$$

(see Lecture Notes 2 and 14 of Astrophysics)

- Case B: conversion of the Lyman series photons.
  - An ionization-bounded nebula with $\tau_0 = 1$ (at $\nu = \nu_0$, Lyc) has $\tau(\text{Ly}\alpha) \approx 10^4$, $\tau(\text{Ly}\beta) \approx 10^3$, $\tau(\text{Ly}8) \approx 10^2$, and $\tau(\text{Ly}18) \approx 10$.
  - In each resonance scattering, therefore, there is a finite probability that the Lyman-line photon will be converted to a lower-series photon plus a lower member of the Lyman series.
  - For instance, when an Lyβ photon is absorbed by an H atom, it is scattered with a probability of $P_{31,10} = 0.882$ or converted into Hα + two photons with $P_{31,20} = 0.118$.



(Here, $P_{nL,n'L'}$ = probability of the transition from n, L to n', L'.)

After ~9 scatterings, an Lyβ photon is converted to Hα + two photons, and cannot escape from the nebula.

  - An Lyγ photon is transformed either into (1) a Paα photon + an Hα photon + an Lyα photon, or (2) into an Hβ photon + two photons ($2\,^2S - 1\,^2S$).
  - Case B: For large optical depths, every Lyman-line photon (if $n \geq 3$) is scattered many times and is converted into lower-series photons plus either Lyα or two-continuum photons. Case B is more accurate than Case A for most nebulae.
  - However, real situation is intermediate, and is similar to Case B for the lower Lyman lines, but approaches to Case A as $n \to \infty$ and $\tau(\text{Ly}n) \to 1$.
  - In Case B, the downward radiative transitions from $n\,^2P^o$ ($n \geq 3$) to $1\,^2S$ are simply omitted from the consideration.

- It is convenient to define the **effective recombination coefficient**:
  - $$n_p n_e \alpha_{nn'}^{\text{eff}} \equiv \sum_{L=0}^{n-1} \sum_{L'=L\pm1} n_{nL} A_{nL,n'L'} = \frac{4\pi j_{nn'}}{h\nu_{nn'}}$$
- For H-like ions of nuclear charge Z,
  - $A_{nL,n'L'} \propto Z^4$. Therefore, $P_{nL,n'L'}$ and $C_{nL,n'L'}$ matrices are independent of Z.
  - $\alpha_{nL}(Z,T) = Z\alpha_{nL}(1,T/Z^2)$, $\alpha_{nn'}^{\text{eff}}(Z,T) = Z\alpha_{nn'}(1,T/Z^2)$
  - $\nu_{nn'}(Z) = Z^2 \nu_{nn'}(1)$
  - Therefore, the emission coefficient is $j_{nn'}(Z,T) = Z^3 j_{nn'}(1,T/Z^2)$
  - Thus, the calculations for H I at a temperature T can be applied to He II at $T' = 4T$:
    $j_{nn'}(2,T') = 8 j_{nn'}(1, T = T'/4)$

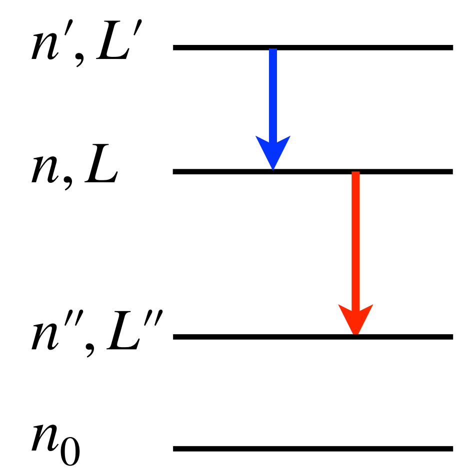
$$\left(E_n = -\frac{Z^2}{n^2}\ \text{Ryd} \right)$$

4.2 Optical Recombination Lines - Collision Effects

- Collision of H with protons
  - Collisions with both electrons and protons can cause the angular-momentum-changing transitions, $nL \rightarrow nL \pm 1$, which have small energy difference. Protons are more effective than electrons because of the slow velocity of protons.
  - These collisional transitions must be included in the equilibrium equations.

$$n_p n_e \alpha_{nL}(T) + \sum_{n'>n}^{\infty} \sum_{L'=L\pm1} n_{n'L'} A_{n'L',nL} + \sum_{L'=L\pm1} n_{nL'} n_p q_{nL',nL}$$

$$= n_{nL} \left[\sum_{n''=n_0}^{n-1} \sum_{L''=L\pm1} A_{nL,n''L''} + \sum_{L''=L\pm1} n_p q_{nL,nL''} \right]$$



where $n_0 = 1$ or 2 for Case A and B, respectively. [Note a typo in Eq. (4.16)]
The collisional transition probability per proton per unit volume is given by

$$q_{nL,n'L'}(T) = \int_0^\infty u\sigma(nL \rightarrow n'L') f(u) du \quad [\mathrm{cm^3\ s^{-1}}].$$

- Thermodynamic Equilibrium between different L states within the same n:
  - For sufficiently large proton densities, the collisional terms dominate, and they tend to set up a TE distribution of the various L levels within each n.
  - Then, the populations are proportional to the statistical weights (because of very tiny energy differences between them):

    $\frac{n_{nL}}{n_{nL'}} = \frac{g_{nL}}{g_{nL'}} = \frac{2L+1}{2L'+1}$ or $n_{nL} = \frac{2L+1}{n^2} n_n \quad \leftarrow \quad \sum_{L=0}^{n-1} (2L+1) = n^2$, where $n_n = \sum_{L=0}^{n-1} n_{nL}$ is the total population in the levels with the same principal quantum number n.
  - As n increases, the collisional cross section $\sigma_{nL\to nL\pm1}$ increases, but the transition probabilities $A_{nL,n'L\pm1}$ decreases. Therefore, the TE condition become increasingly good approximations with increasing n.
  - The typical cross sections for protons at $T \approx 10^4$ K are

$$\sigma(2\,^2S \to 2\,^2P^o) \approx 3\times10^{-10}\text{cm}^2$$

$$\sigma(10\,^2L \to 10\,^2L\pm1) \approx 4\times10^{-7}\text{cm}^2$$

$$\sigma(20\,^2L \to 20\,^2L\pm1) \approx 6\times10^{-6}\text{cm}^2$$

  - **There is a level n_{cL} above which the TE applies.** At $T \approx 10{,}000$ K, they are

$$n_{cL} \approx 15 \text{ at } n_p \approx 10^4\,\text{cm}^{-3}$$

$$n_{cL} \approx 30 \text{ at } n_p \approx 10^2\,\text{cm}^{-3}$$

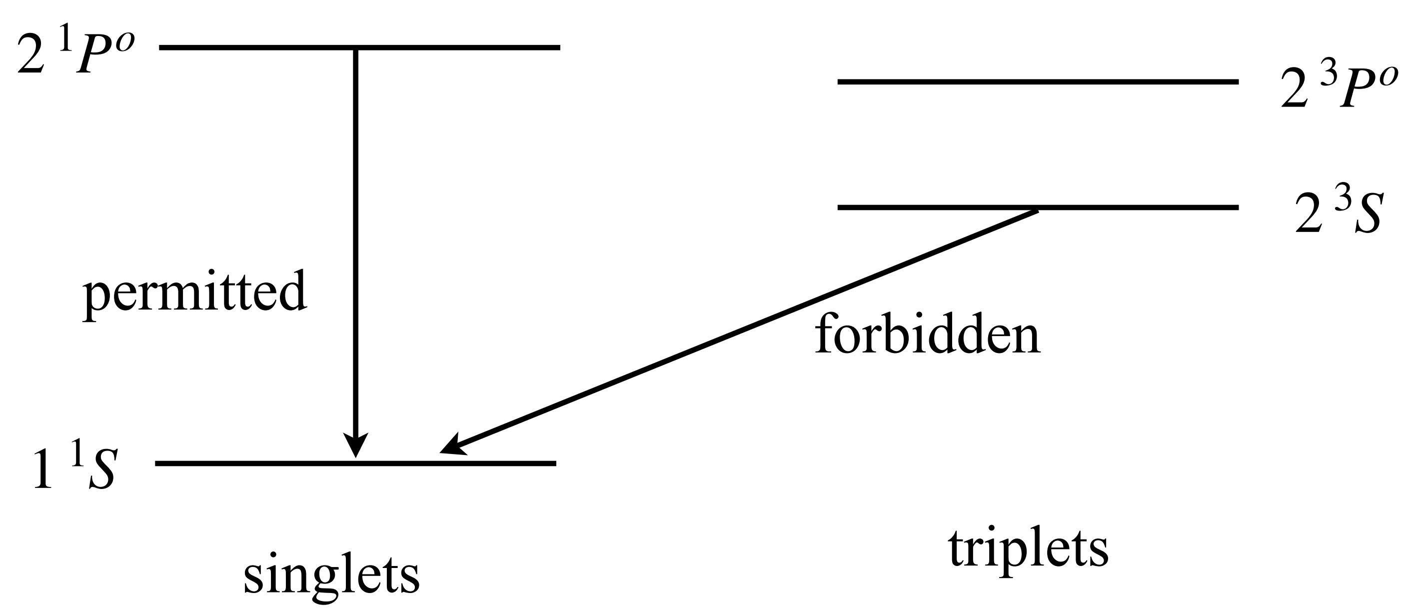
$$n_{cL} \approx 45 \text{ at } n_p \approx 1\,\text{cm}^{-3}$$

- The same type of effect occurs in the He II spectrum.
  - The He II lines are emitted in the H^+ and He^{++} zone, so both H^+ ions (protons) and He^{++} ions (α particles) can cause collisional, angular momentum-changing in the excited levels of He^+. The cross sections $\sigma_{nL\to nL\pm1}$ are larger for the He^{++} ions than for the H^+ ions. Both of them must be taken into account in the He^{++} region.
  - The level n_{cL} above which the TE condition can apply for He II is

    $n_{cL} \approx 22$ at $n_p \approx 10^4\,\mathrm{cm}^{-3}$, $n_{cL} \approx 32$ at $n_p \approx 10^2\,\mathrm{cm}^{-3}$ when $T \approx 10{,}000$ K.
- $\sigma(n, L \to n, L \pm 1) \gg \sigma(n, L \to n \pm 1, L \pm 1)$.
  - For the transitions $(n, L \to n \pm 1, L \pm 1)$, collisions with electrons are more effective than collisions with protons.
  - At $T \approx 10^4$ K, the representative cross sections for electrons are of order $\sigma(nL \to n \pm \Delta n, L \pm 1) \approx 10^{-16}\,\mathrm{cm}^2$
  - The effects of these collisions can be incorporated into the equilibrium equations.
  - The cross sections decreases with increasing Δn (but not too rapidly). Collisions with $\Delta n = 1, 2, 3, \cdots$ must all be included in the equilibrium equations.
- The computational work required to set up and solve the equations becomes increasingly complicated and lengthy, but is straightforward in principle.
- Collisions tend to couple levels with $\Delta L = \pm 1$ and small Δn. This coupling increases with increasing n_e (and n_p) and with increasing n.
  - With collisions taken into account, the b_{nL} factors and resulting emission coefficients are no longer independent of density.
  - Table 4.4 (for H I) and Table 4.5 (for He II) shows that the density dependence is rather small.

- Exactly the same formalism can be applied to He I recombination lines.
  - The singlet and triplets can be treated as approximately separate systems.
  - The He I triplets always follow Case B, because downward radiative transitions to $1\,^1S$ (singlet) essentially do not occur.
  - For the singles, Case B is usually a better approximation than Case A.
  - He I $1\,^1S - n\,^1P^o$ line photons can photoionize H^0, and thus may be destroyed before they are converted into lower-energy photons.

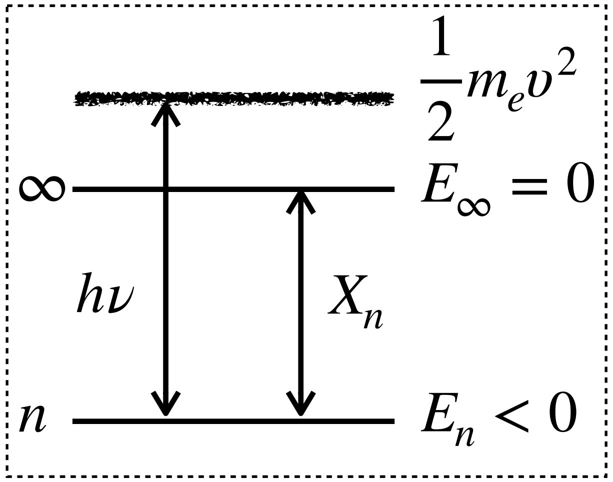
selection rule: $\Delta S = 0$



4.3 Optical Continuum Radiation

- Continuum
  - In addition to the recombination lines in the bound-bound transitions, recombination processes also lead to the relatively weak emission in free-bound and free-free transition.
  - The H I continuum is the strongest. The He II continuum may also be significant if He is mostly doubly ionized, but the He I continuum is always weaker.
  - In the optical region, the free-bound continua are stronger. But, in the IR and radio regions the free-free continuum dominates.
  - There is also the two-photon decay of the $2\,^2S$ level of H.
- Free-bound continuum
  - Suppose the H I free-bound continuum radiation at frequency ν, resulting from recombination of free electrons with velocity u to levels with quantum number $n \geq n_1$ and ionization potential X_n, where

$$h\nu = \frac{1}{2}u^2 + X_n \text{ and } h\nu \geq X_{n_1} = \frac{h\nu_0}{n_1^2}$$



  - Its emission coefficient is $j_\nu = \frac{1}{4\pi} n_p n_e \sum_{n=n_1}^{\infty} \sum_{L=0}^{n-1} u\sigma_{nL}(\mathrm{H}^0, u) f(u) h\nu \frac{du}{d\nu}$.
  - The recombination cross sections can be calculated from the photoionization cross sections using the Milne relation.

- Free-free (or bremsstrahlung) continuum
  - The emission coefficient emitted by positive ions of charge Z is

$$j_\nu = \frac{1}{4\pi} n_+ n_e \frac{32 Z^2 e^4 h}{3 m_e^2 c^3} \left(\frac{\pi h \nu_0}{3kT} \right)^{1/2} \exp\left(-h\nu/kT\right) g_{\rm ff}(T, Z, \nu), \quad h\nu_0 = R_\infty = \frac{2\pi^2 m_e e^4}{ch^3}$$

where $g_{\rm ff}(T, Z, \nu) \approx 1 - 5$ is a Gaunt factor. (see Lecture note 8 of Astrophysics)

- Free-bound + free-free
  - The emission coefficient for the H I recombination continuum, including both free-bound and free-free contributions, may be written

$$j_\nu(\text{H I}) = \frac{1}{4\pi} n_p n_e \gamma_\nu(\text{H}^0, T)$$

  - The contributions to the continuum-emission coefficient from He I and He II may be written

$$j_\nu(\text{He I}) = \frac{1}{4\pi} n(\text{He}^+) n_e \gamma_\nu(\text{He}^+, T), \quad \text{and} \quad j_\nu(\text{He II}) = \frac{1}{4\pi} n(\text{He}^{++}) n_e \gamma_\nu(\text{He}^{++}, T)$$

  - The numerical values of $\nu\gamma_\nu$ are shown in Table 4.7, 4.8 and 4.9, and Figure 4.1.
  - The calculation for He II is exactly analogous to that for H I.
  - But, for He I, there is no L degeneracy.

- **Two-photon continuum-emission**
  - The transition probability for the two-photon decay is $A_{2\,^2S\to 1\,^2S} = 8.23\ \mathrm{s}^{-1}$.
  - The sum of the energies of two photons is $h\nu' + h\nu'' = h\nu_{12} = h\nu_{\mathrm{Ly}\alpha} = (3/4)h\nu_0$.
  - The probability distribution of the emitted photons is therefore symmetric around the frequency $(1/2)\nu_{12} = 1.23 \times 10^{15}\ \mathrm{s}^{-1}$, corresponding to $\lambda = 2431$ Å. The emission coefficient in this two photon continuum may be written

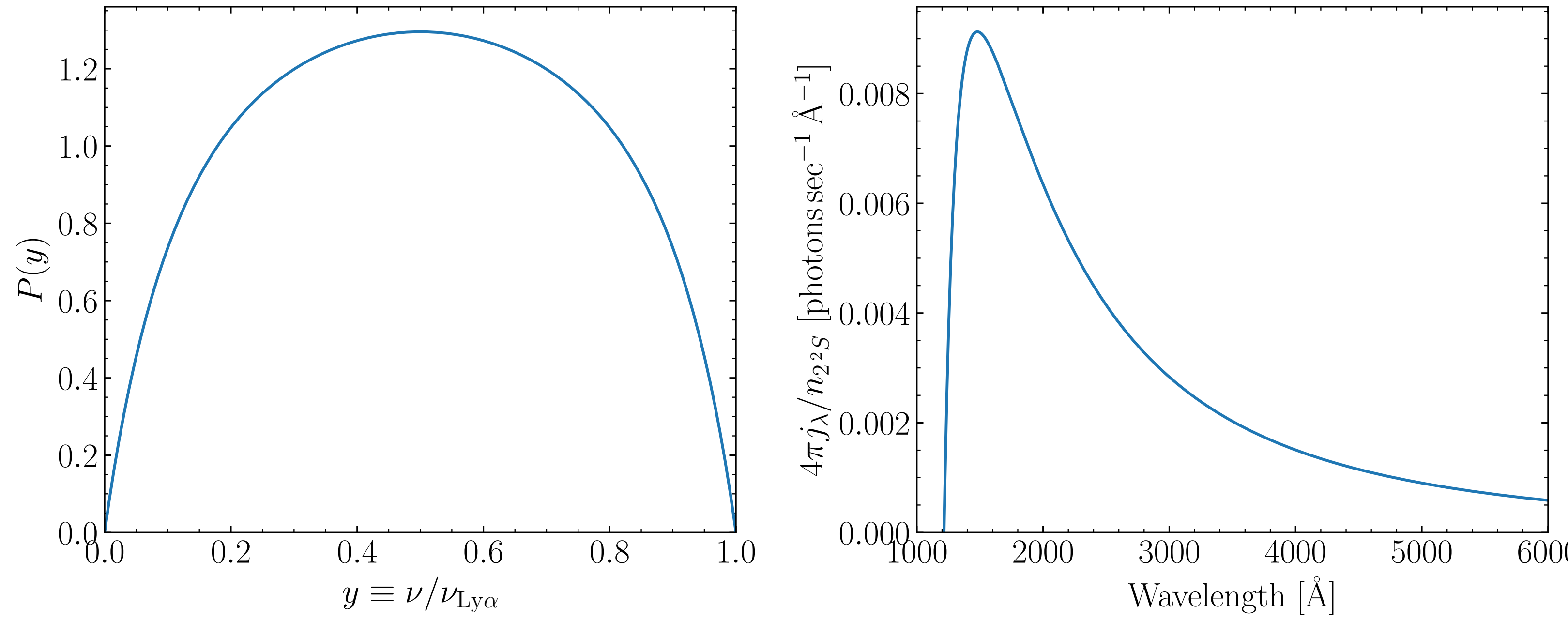
$$j_\nu(2q) = \frac{1}{4\pi} n_{2\,^2S} A_{2\,^2S,1\,^2S} \frac{h\nu}{\nu_{12}} P(y) \quad \Rightarrow \quad \int j_\nu d\nu = \frac{1}{4\pi} n_{2\,^2S} A_{2\,^2S,1\,^2S} \int_0^1 h\nu P(y) dy$$

$$y = \frac{\nu}{\nu_{12}} \text{ and } \overline{P}(\nu) = \frac{1}{\nu_{12}} P(y) \quad \text{[Note a typo in Eq. (4.25)]}$$

Here, $P(y)dy$ is the normalized probability per decay that one photon is emitted in the range of frequencies $y\nu_{12}$ to $(y+dy)\nu_{12}$.

The spectral shape is symmetric about $\nu_{12}/2$ if expressed in photons per unit frequency.

However, it is not symmetric about $\lambda 2431$ if expressed per unit wavelength.



- To express the two-photon continuum-emission coefficient, we need to calculate the equilibrium population of $n(2\,^2S)$.
- In low-density nebulae, the equilibrium is given by $\boxed{n_p n_e \alpha^{\rm eff}_{2\,^2S}({\rm H}^0, T) = n_{2\,^2S} A_{2\,^2S \to 1\,^2S},}$

  where $\alpha^{\rm eff}_{2\,^2S}$ is the effective recombination rate coefficient for populating $2\,^2S$ by direct recombinations and by recombinations to higher levels followed by cascade to $2\,^2S$.
- At finite densities, angular-momentum-changing transition from $2\,^2S$ to $2\,^2P^o$ by collisions with protons and electrons reduce the population of $2\,^2S$. The protons are more effective than electrons (but electrons are not completely negligible). With these collisional processes taken into account, the population in $2\,^2S$ is given by

$$\boxed{n_p n_e \alpha^{\rm eff}_{2\,^2S}({\rm H}^0, T) = n_{2\,^2S}\left(A_{2\,^2S,1\,^2S} + n_p q_{2\,^2S,2\,^2P^o} + n_e q_{2\,^2S,2\,^2P^o}\right)}$$

$$j_\nu(2q) = \frac{1}{4\pi} n_{2S} A_{2S,1S}\, h\nu \overline{P}(\nu) = \frac{1}{4\pi} n_p n_e \gamma_\nu(2q), \;\; \text{where} \;\; \gamma_\nu(2q) = \frac{\alpha^{\rm eff}_{2\,^2S}\, h\nu \overline{P}(\nu)}{1 + \left[\dfrac{n_p q^p_{2S,2P} + n_e q^e_{2S,2P}}{A_{2S,1S}}\right]}$$

$g_\nu = h\nu\overline{P}(\nu)$ is the frequency dependence of ${\rm H}^0$ two-photon emission coefficient.

Note that the symmetry $\overline{P}(\nu) = \overline{P}(\nu' = \nu_{12} - \nu)$ and thus $g_\nu = (\nu/\nu')\, g_{\nu'}$

- **Collisional deexcitation of $2\,^2S$ via $2\,^2P^o$ is more important than two-photon decay for $n_p \geq 10^4\,{\rm cm}^{-3}$.**

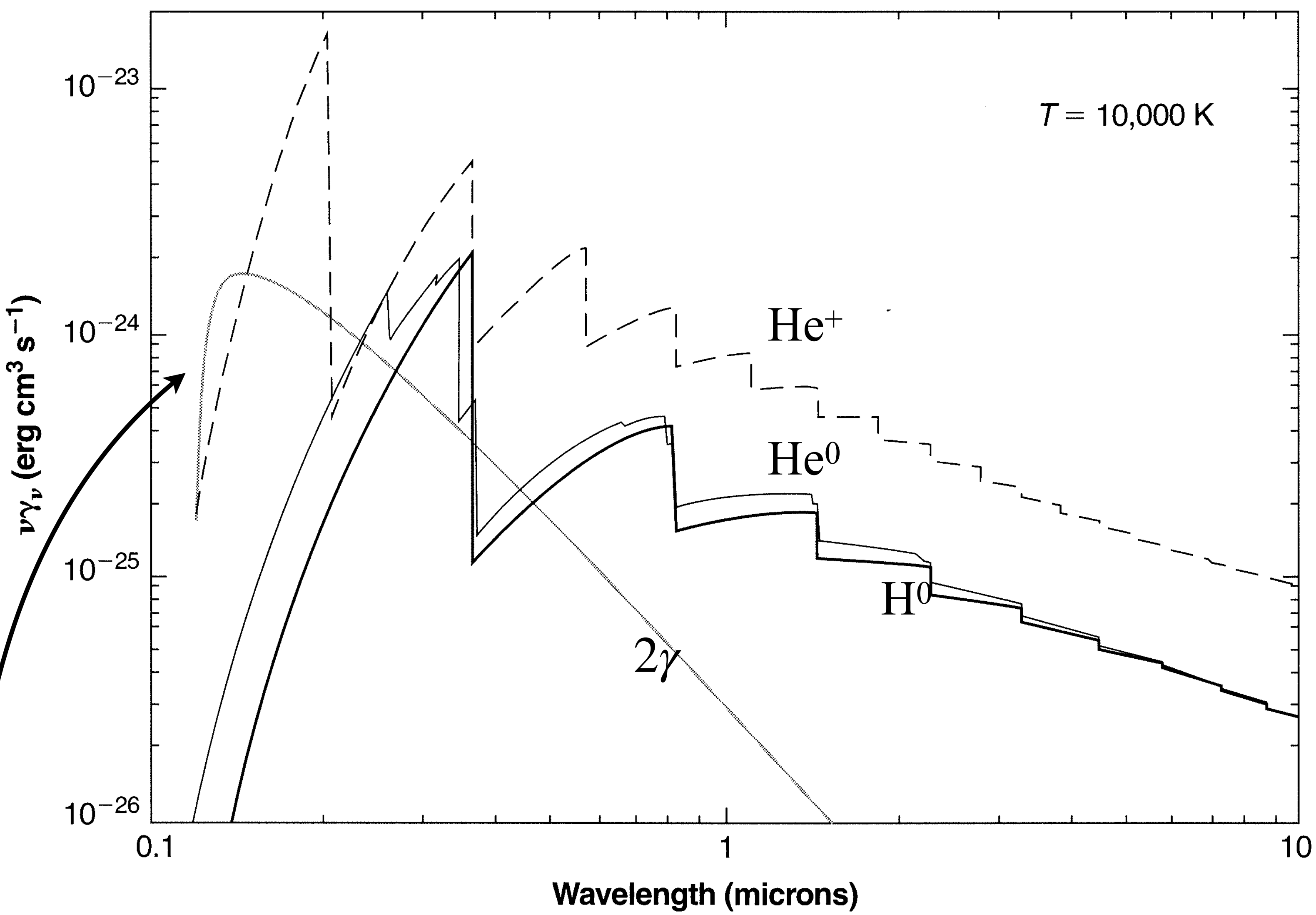


Figure 4.1
Frequency variation of continuous-emission coefficient γ_ν(H^0, solid line), γ_ν (He^0, thin solid line), γ_ν (He^+, dashed line), and γ_ν ($2h\nu$, smooth solid line) in the low-density limit $n_e \to 0$, all at $T = 10{,}000$ K.

The figure shows the large discontinuities at the ionization potentials of the various excited level.

For a He abundance of 10%, if the He is mostly doubly ionized, then the He II contribution is roughly comparable to that of H I.

But, if the He is mostly singly ionized, the He I contribution is only about 10% of the H I contribution.

Two photon emission is significant in comparison with the H I continua, just above the Balmer limit at λ3646Å.

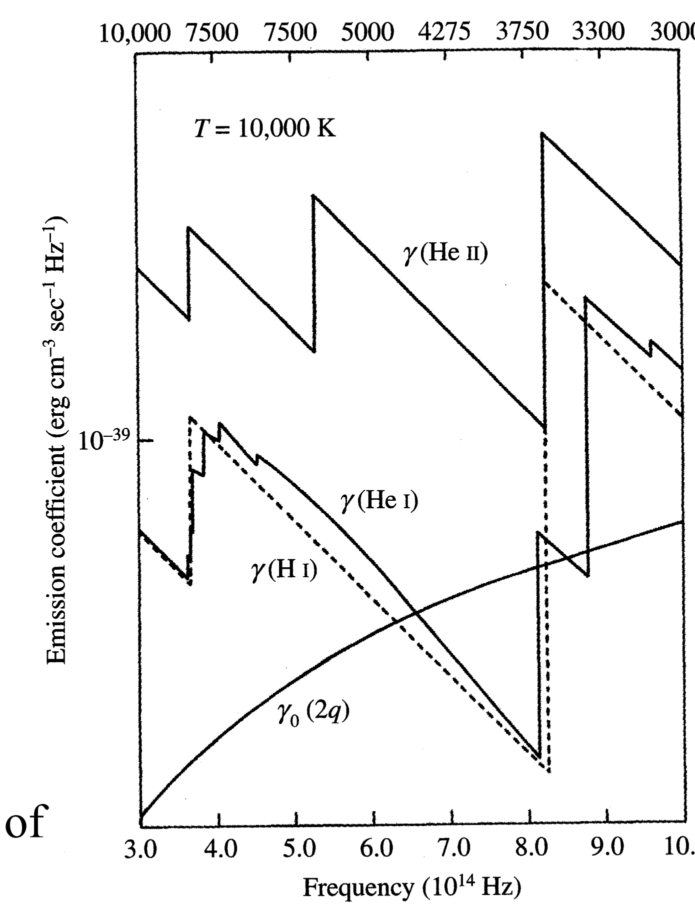


Figure 6.1 in Physics and Chemistry of the Interstellar Medium (Sun Kwok)

4.4 Radio-Frequency Continuum and Line Radiation (1) Continuum

- In the radio-frequency region $h\nu \ll kT$ and thus stimulated emission is much more important than in the optical region.
- **Free-free emission:** The radio-frequency continuum is due to free-free emission. The emission coefficient is the same as that applies in the optical region.

$$j_\nu = \frac{1}{4\pi} n_+ n_e \frac{2^5 \pi e^6 Z^2}{3 m_e c^3} \left(\frac{2\pi}{3 m_e k T} \right)^{1/2} \exp\left(-h\nu/kT\right) g_{\mathrm{ff}}(T, Z, \nu),$$

where $g_{\mathrm{ff}}(T, Z, \nu) = \frac{\sqrt{3}}{\pi} \left[\ln \left(\frac{8 k^3 T^3}{\pi^2 Z^2 e^4 m \nu} \right) - \frac{5\gamma}{2} \right]$, $\gamma = 0.577$ is Euler's constant.

Numerically, this is approximately

$$g_{\mathrm{ff}}(T, Z, \nu) = \frac{\sqrt{3}}{\pi} \left(\ln \frac{T^{3/2}}{Z\nu} + 17.7 \right)$$ with T in K and ν in Hz.

At $T \approx 10^4$ K, $\nu \approx 10^3$ MHz, $g_{\mathrm{ff}} \approx 10$.

- **Free-free absorption:** The free-free "effective" absorption coefficient is found from Kirchhoff's law.
  - $\kappa_\nu = j_\nu / B_\nu$ where $B_\nu = (2h\nu^3/c^2)\left[\exp\left(h\nu/kT\right) - 1\right]^{-1} \approx 2\nu^2 kT/c^2$ if $h\nu/kT \ll 1$

    $$\therefore \kappa_\nu = n_+ n_e \frac{16\pi^2 Z^2 e^6}{(6\pi m_e kT)^{3/2}\nu^2 c} g_{\rm ff} \quad \text{per unit length.}$$

    This effective absorption coefficient is the difference between the true absorption and the stimulated emission. $1 - \exp(-h\nu/kT) \approx h\nu/kT \ll 1$.
  - The optical depth is obtained as follows, after substituting numerical values and fitting powers to the weak temperature and frequency dependence of $g_{\rm ff}$:

    $$\begin{aligned} \tau_\nu &= \int \kappa_\nu ds \\ &= 8.24 \times 10^{-2} T^{-1.35} \nu_9^{-2.1} \int n_+ n_e ds \\ &= 8.24 \times 10^{-2} T^{-1.35} \nu_9^{-2.1}\, \mathrm{EM_c/cm^{-6}pc} \end{aligned}$$

    where T is measured in K, $\nu_9 = \nu/10^9$ Hz, and $\mathrm{EM_c}$ is the continuum emission measure in units of $\mathrm{cm^{-6}pc}$.
  - From the optical depth, **all nebulae become optically thick at low frequencies, optically thin at high frequencies.**
    - $\tau_\nu \approx 1$ at $\nu \approx 200$ MHz for an H II region with $n_e \approx n_p \approx 10^2\,\mathrm{cm^{-3}}$ and a diameter 10 pc.
    - $\tau_\nu \approx 1$ at $\nu \approx 600$ MHz for a planetary nebula with $n_e \approx 3 \times 10^3\,\mathrm{cm^{-3}}$ and a diameter 0.1 pc.

- The equation of RT

$$\frac{dI}{ds} = -\kappa_\nu I_\nu + j_\nu \quad \Rightarrow \quad \frac{dI}{d\tau_\nu} = -I_\nu + \frac{j_\nu}{\kappa_\nu} = -I_\nu + B_\nu(T) \textbf{ in LTE}.$$

- It there is no incident radiation, the solution is given by $I_\nu = \int_0^{\tau_\nu} B_\nu(T)\exp(-\tau_\nu)d\tau_\nu$.

- In the radio-frequency region, $B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/kT)-1} \approx \frac{2\nu^2 kT}{c^2}$.

- It is conventional in radio astronomy to measure intensity in terms of brightness temperature, defined by $T_{b\nu} = c^2 I_\nu / 2\nu^2 k$. Then, the RT equation becomes

$$T_{b\nu} = \int_0^{\tau} T\exp(-\tau_\nu)d\tau_\nu$$

- For an isothermal nebula, the solution becomes

$$T_{b\nu} = T(1-e^{-\tau_\nu}) \begin{cases} T\tau_\nu & \text{as } \tau_\nu \to 0 \\ T & \text{as } \tau_\nu \to \infty \end{cases}$$

Therefore, $T_{b\nu} \propto \nu^{-2}$ at high frequency and $T_{b\nu}$ = constant at low frequency.

4.4 Radio-Frequency Continuum and Line Radiation (2) Line

- The H I recombination lines of very high n belong to the radio-frequency spectral region.
  - Observed examples:
    - H 109α (from $n = 110$ to $n = 109$) at $\nu = 5008.89$ MHz, $\lambda = 5.99$ cm
    - H 137β (from $n = 139$ to $n = 137$) at $\nu = 5005.0$ MHz, $\lambda = 6.00$ cm
- For all line observed in the radio-frequency region, $n > n_{cL}$ (**a level above which the TE can be applied**), so that $n_{nL} \propto (2L+1)$ at a fixed n, and only the populations n_n need be considered.
- Collisional ionization of levels with large n and its inverse process (three-body recombination) must also be taken into account in the equations of statistical equilibrium of level populations.

$$\mathrm{H}^0(n) + e \Longleftrightarrow \mathrm{H}^+ + e + e$$

  - The **rate of collisional ionization** per unit volume per unit time from level n is $n_n n_e \left\langle u\sigma_{\rm c.i.}(n) \right\rangle = n_n n_e q_{n,i}(T)$, where $q_{n,i} = \left\langle u\sigma_{\rm c.i.} \right\rangle$ is the collisional ionization rate coefficient.
  - The **rate of three-body recombination** per unit volume per unit time may be written $n_p n_e^2 \phi_n(T)$. From the principle of detailed balancing, the three-body recombination rate coefficient is obtained: $\phi_n(T) = n^2 \left(\dfrac{h^2}{2\pi m_e kT} \right)^{3/2} \exp\left(X_n/kT \right) q_{n,i}(T)$

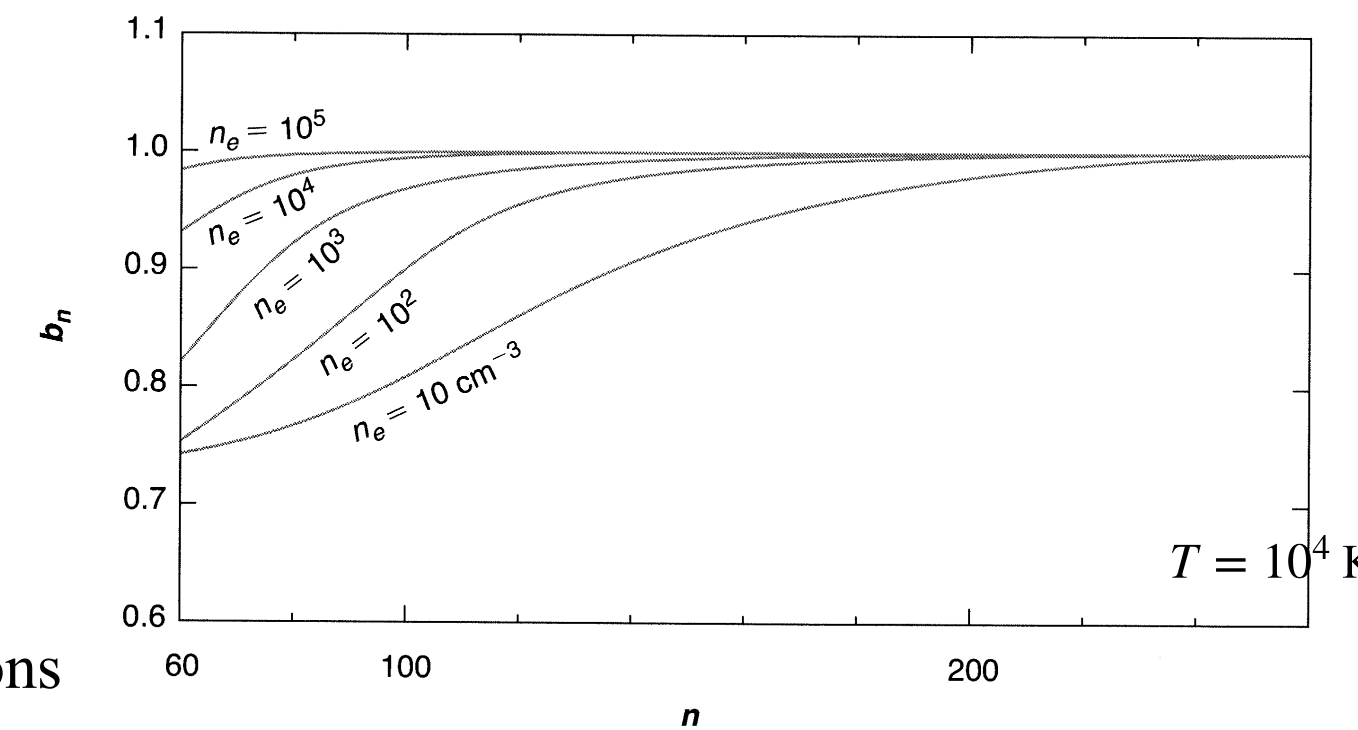
- Then, the equilibrium equation at high n becomes

$$n_p n_e \left[\alpha_n(T) + n_e \phi_n(T)\right] + \sum_{n'>n}^{\infty} n_{n'} A_{n',n} + \sum_{n'=n_0}^{\infty} n_{n'} n_e q_{n',n} = n_n \left[\sum_{n'=n_0}^{n-1} A_{n,n'} + \sum_{n'=n_0}^{\infty} n_e q_{n,n'}(T) + n_e q_{n,i}(T) \right]$$

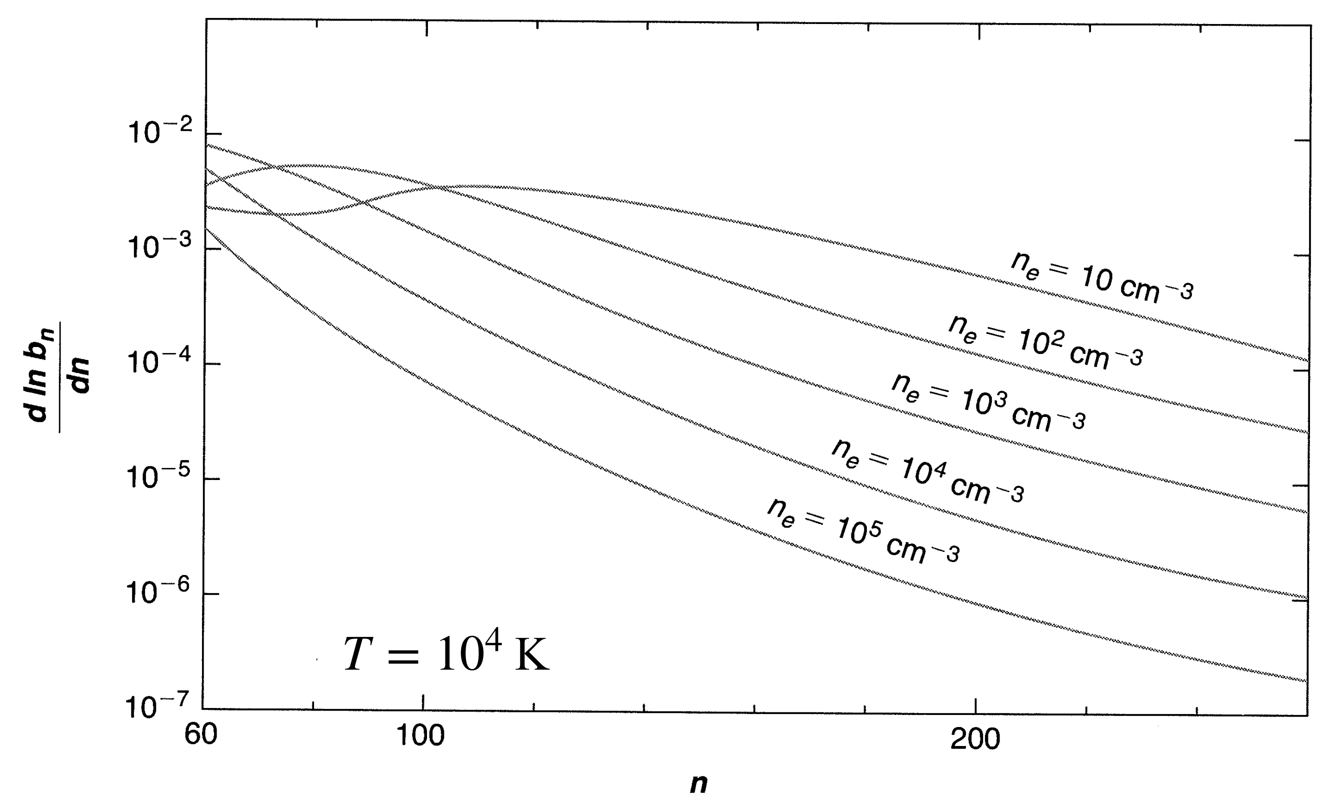
radiative rec. + 3-body rec. + radiative decay + collisional deexcitation = radiative decay + collsional deexcitation + collisional ionization

where $A_{n,n'} = \frac{1}{n^2} \sum_{L,L'} (2L+1) A_{nL,n'L'}$ is the mean transition probability averaged over all the L levels of the upper principal quantum number.

- These equations can be expressed in terms of b_n instead of n_n, and the solutions can be found numerically by matrix-inversion techniques.
  - Since the b_n factors have been defined w.r.t. thermodynamic equilibrium at T, n_e, and n_p, the coefficient b_∞ for the free electrons is identically unity ($b_\infty = 1$).
  - Figure 4.2 (upper panel) shows that the increasing importance of collisional transitions as n_e increases makes $b_n \approx 1$ at lower and lower n.



- Radiative Transfer and Stimulated Emission
  - To calculate the emission in a specific recombination line, it is necessary to solve the equation of RT, taking account of the effects of stimulated emission.
  - If $k_{\nu l}$ is the true line-absorption coefficient, the line-absorption coefficient corrected for stimulated emission is
  - $$k_{\nu L} = k_{\nu l}\left(1 - \frac{n_m/g_m}{n_n/g_n}\right) = k_{\nu l}\left[1 - \frac{b_m}{b_n}\exp\left(-h\nu_{mn}/kT\right)\right] \quad \text{(see Astrophysics Lecture 2)}$$
  - Since $b_m/b_n \approx 1$ and $h\nu \ll kT$, we can expand it in a power series and obtain
  - $$k_{\nu L} = k_{\nu l}\left[\frac{b_m}{b_n}\frac{h\nu}{kT} - \frac{d\ln(b_n)}{dn}\Delta n\right] \quad \Leftarrow \quad b_m \approx b_n + \frac{db_n}{dn}\Delta n \text{ and } e^{-h\nu/kT} \approx 1 - h\nu/kT$$
  - This coefficient can become negative, implying maser action, if $(d\ln b_n)/dn$ is sufficiently large.
  - Since $h\nu/kT \approx 10^{-5}$ for typical observed lines, Figure 4.2 shows that this is often the case and the maser effect is in fact often quite important.



4.5 Radiative Transfer Effects in H I

- Optical Thickness and Resonance Lines
  - In most lines, the nebulae are optically thin, and they simply escapes.
  - However, in some lines, especially the resonance lines of abundant atoms, the optical depths are appreciable, and scattering and absorption must be taken into account in calculating the expected line strengths.
- Two extreme assumptions
  - Case A: a nebula with vanishing optical thickness in all the H I Lyman lines
  - Case B: a nebula with large optical depths in all the Lyman lines.
  - These two cases do not require a detailed radiative-transfer solution.
  - In the intermediate cases, a more sophisticated treatment is necessary.
- Other RT problems arise in connection with (1) the He I triplets, (2) the conversion of He II Lyα and H I Lyβ into observable O III or O I line radiation, respectively, by the Bowen resonance-fluorescence precesses, and (3) fluorescence excitation of other lines by stellar continuum radiation.

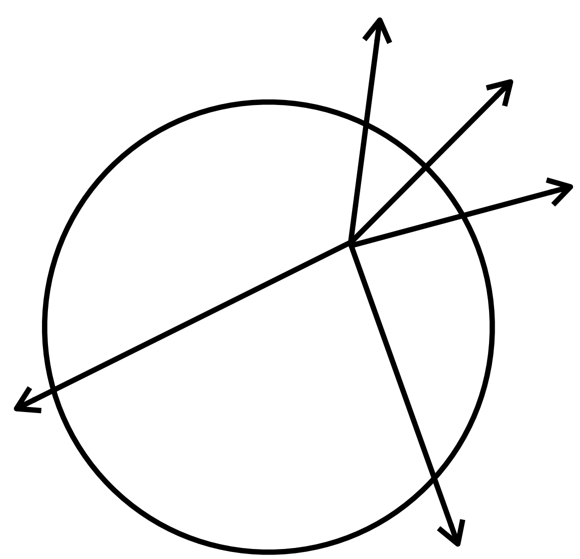
- In a static nebula, the only line-broadening mechanisms are thermal Doppler broadening and radiative damping. In the cores, where radiative damping can be neglected, the line-absorption coefficient has the Doppler form (Gaussian):
- $k_{\nu l} = k_{0l} \exp\left[-\left(\Delta\nu/\Delta\nu_D\right)^2\right] = k_{0l} \exp\left(-x^2\right) \quad [\mathrm{cm}^2]$

Here, l indicates that the coefficient or optical depth is for line.

- $k_{0l} = \dfrac{\sqrt{\pi} e^2 f_{ij}}{m_e c \Delta\nu_D}$ is the line-absorption cross section at the center of the line,

  $\Delta\nu_D = \sqrt{\dfrac{2kT}{m_{\mathrm{H}} c^2}}\nu_0 = \dfrac{v_{\mathrm{th}}}{c}\nu_0$ [Hz] is the thermal Doppler width (Hz), $\Delta\nu = \nu - \nu_0$, and f_{ij} is the absorption oscillator strength between the lower level (i) and upper level (j). The full-width at half-maximum (FWHM) of the line is $2\sqrt{\ln 2}\,\Delta\nu_D$.
- Small-scale micro-turbulence can be taken into account as a further source of broadening of the line-absorption coefficient by adding the thermal and turbulent velocity terms in quadrature, $\Delta\nu_D^2 \rightarrow b^2 = \Delta\nu_{\mathrm{thermal}}^2 + \Delta\nu_{\mathrm{turbulent}}^2$.
- Large scale turbulent and expansion of the nebula can be treated by considering the frequency shift between the emitting and absorbing volumes (e.g. Sololev approximation or large gradient velocity (LGV) method).

- Escape probability
  - In a static nebula, a photon emitted at a particular point in a particular direction and with a normalized frequency $x = (\nu - \nu_0)/\Delta\nu_D$ has a probability $\exp(-\tau_x)$ of escaping from the nebula without further scattering and absorption. Here, τ_x is the optical depth from the point to the edge of the nebula in this direction and at this frequency.
  - Averaging over all directions gives the mean escape probability from this point and at this frequency.
  - Further averaging over the frequency profile of the emission coefficient gives the mean escape probability from the point.
  - For all the forbidden lines and for most of the other lines, the optical depths are so small in every direction, even at the center of the line, that the mean escape probabilities from all points are essentially unity.
  - However, for lines of larger optical depth we must examine the probability of escape quantitatively.



$$\langle P_{\rm esc} \rangle = \iiint \left[1 - e^{-\tau_\nu(\mathbf{r},\Omega)}\right] d\Omega dV d\nu$$

$\tau_\nu(\mathbf{r}, \Omega)$ = optical depth from a point $\mathbf{r}$ to the boundary measured in a direction Ω.

average over all directions (4π), over all points within the medium, and over all frequencies

- Escape probability in a spherical nebula
  - Consider a homogeneous spherical nebula with optical radius τ_0 in the line center.
  - If, at a particular normalized frequency x, the optical radius of the nebula is τ_x, the mean escape probability averaged over all directions and volumes is

$$p(\tau_x) = \frac{3}{4\tau_x}\left[1 - \frac{1}{2\tau_x^2} + \left(\frac{1}{\tau_x} + \frac{1}{2\tau_x^2}\right)\exp(-2\tau_x)\right]$$

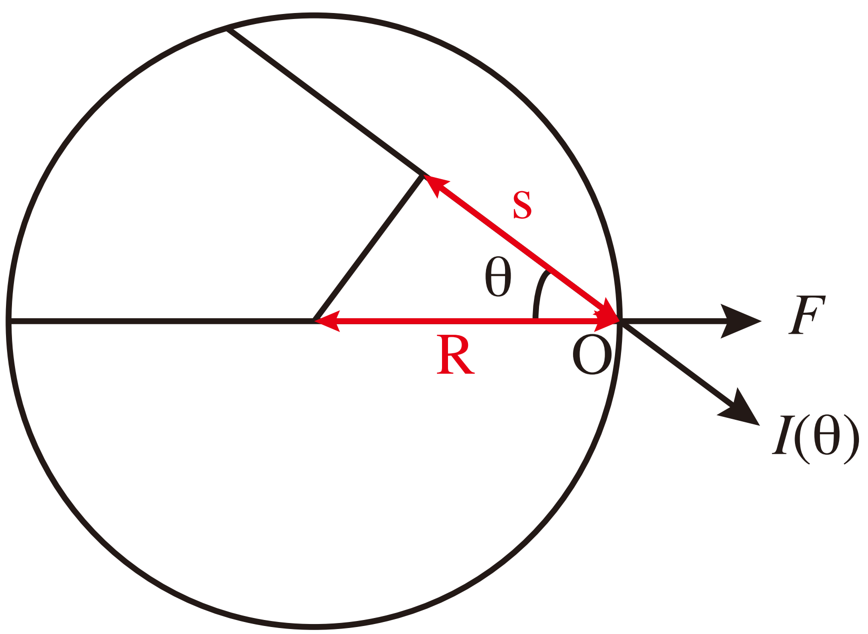
  - If $\tau_0 < 10^4$, only the Doppler core of the line absorption cross section need be considered. In this case, when we average over the Doppler profile, the mean escape probability for a photon emitted in the line is

$$\varepsilon(\tau_0) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} p(\tau_x)\exp(-x^2)dx$$

  - This integral must be evaluated numerically, but for $\tau_0 \leq 50$, the results can be fitted fairly accurately with

$$\varepsilon(\tau_0) = \frac{1.72}{\tau_0 + 1.72}$$

Derivation of the escape probability formula



$s = R\cos\theta \qquad \tau_0 = \kappa R$

a homogeneous sphere with
a constant emission coefficient j
a constant absorption coefficient κ
no external source

[No absorption]

Intensity along the θ direction at the surface (point O)

$$I(\theta) = \int_{-s}^{s} j dl = 2js = 2jR\cos\theta$$

Flux at the surface

$$F_0 = \int_0^{2\pi}\int_0^{\pi/2} I(\theta)\cos\theta\sin\theta d\theta d\phi = (4\pi jR)\int_0^1 \mu^2 d\mu$$

$$= \frac{4\pi}{3} jR$$

[Absorption]

From the RT equation solution

$$I(\theta) = S\left[1 - e^{-\tau(\theta)}\right]$$

Here, $S = j/\kappa$ is the source function, and $\tau(\theta) = 2R\cos\theta\kappa$ is the optical depth along the θ direction

Escaping flux at the surface

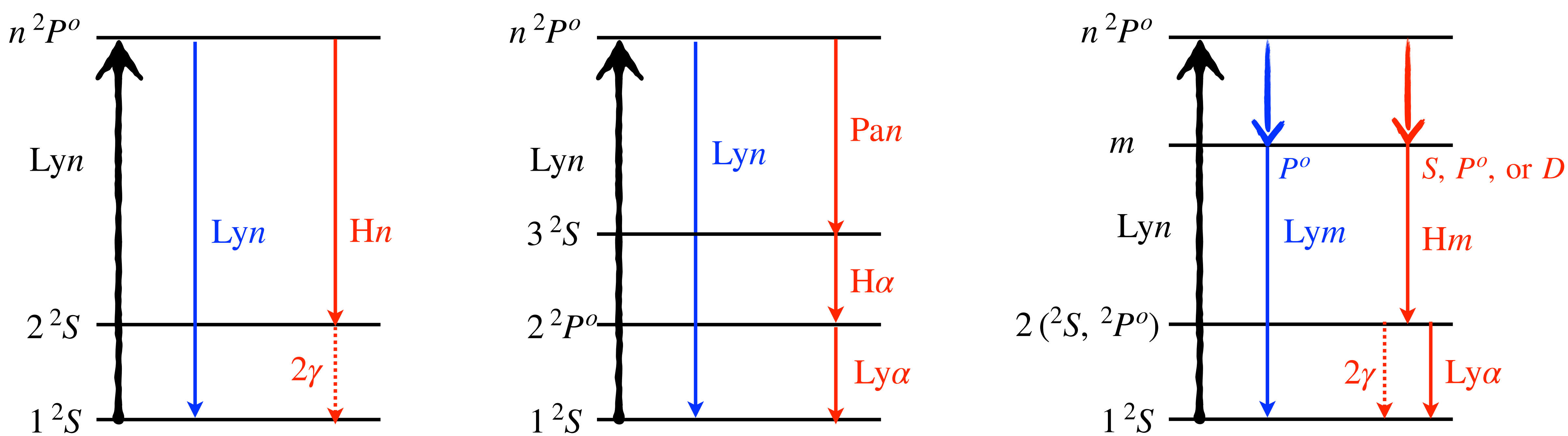
$$F_{\rm esc} = 2\pi\int_0^1 I(\theta)\mu d\mu = \frac{2\pi j}{\kappa}\int_0^1 \left(1 - e^{-2\tau_0\mu}\right)\mu d\mu$$

$$= \frac{\pi j}{\kappa}\left[1 + \left(\frac{1}{\tau_0} + \frac{1}{2\tau_0^2}\right)e^{-2\tau_0} - \frac{1}{2\tau_0^2}\right]$$

Therefore, the **escape probability** is

$$f_{\rm esc} = \frac{F_{\rm esc}}{F_0} = \frac{3}{4\tau_0}\left[1 + \left(\frac{1}{\tau_0} + \frac{1}{2\tau_0^2}\right)e^{-2\tau_0} - \frac{1}{2\tau_0^2}\right]$$

- Lyman lines : resonance scattering or resonance florescence
  - A Lyman line Lyn can be absorbed by another hydrogen atom, and each absorption process represents an excitation of the $n\,^2P^o$ level of H^0.
  - This excited level very quickly undergoes a radiative decay. The result is either resonance scattering or resonance fluorescence excitation of another H I line.



  - Let's define $P_n(\text{Ly}m)$ and $P_n(\text{H}m)$ as the probability that absorption of an Lyn photon results in emission of an Lym photon and of an Hm photon, respectively. Then, they can be calculated from the probability and cascade matrices. (see the rightmost panel in the above figure)

$$P_n(\text{Ly}m) = C_{n1,m1}P_{m1,10}$$

$$P_n(\text{H}m) = C_{n1,m0}P_{m0,21} + C_{n1,m1}P_{m1,20} + C_{n1,m2}P_{m2,21}$$

$$P \Rightarrow S \rightarrow P \qquad P \Rightarrow P \rightarrow S \qquad P \Rightarrow D \rightarrow P$$

Here, $0 = S$
$1 = P^o$
$2 = D$

- **Calculation of the emergent Lyman-line spectrum** using these probabilities
  - R_n = total number of Lyn photons generated per unit time by recombination and subsequent cascading
  - A_n = total number of Lyn photons absorbed per unit time
  - J_n = total number of Lyn photons emitted per unit time = sum of the contributions from recombination and from resonance fluorescence plus scattering:

$$J_n = R_n + \sum_{m=n}^{\infty} A_m P_m(\mathrm{Ly}n) \qquad \text{Eq(1)}$$

  - ε_n = escape probability of individual Lyn photon. The total number of Lyn photons escaping per unit time is

$$E_n = \varepsilon_n J_n = \varepsilon_n \left[R_n + \sum_{m=n}^{\infty} A_m P_m(\mathrm{Ly}n) \right] \qquad \text{Eq(2)}$$

  - In a steady state, the number of Lyn photons emitted per unit time = the numbers absorbed + the number escaping per unit time:

$$J_n = A_n + E_n = A_n + \varepsilon_n J_n \qquad \text{Eq(3)}$$

  - Eliminating J_n in equations (1) and (3), $\boxed{A_n = (1-\varepsilon_n)\left[R_n + \sum_{m=n}^{\infty} A_m P_m(\mathrm{Ly}n) \right]}$ Eq(4)
  - R_n and $P_m(\mathrm{Ly}n)$ are known from the recombination theory. ε_n are known from the RT theory. We can solve the equation for A_n, working downward from the highest n at which ε_n differs appreciably from unity. ($f_{1n} \to 0$ and $\varepsilon_n \to 1$ as $n \to \infty$, see next page).
  - Then, the E_n may be calculated from Eq (2), giving the emergent Lyman-line spectrum.

Oscillator strength of Lyman- and Balmer-lines

Eq. (5.101) Sun Kwok

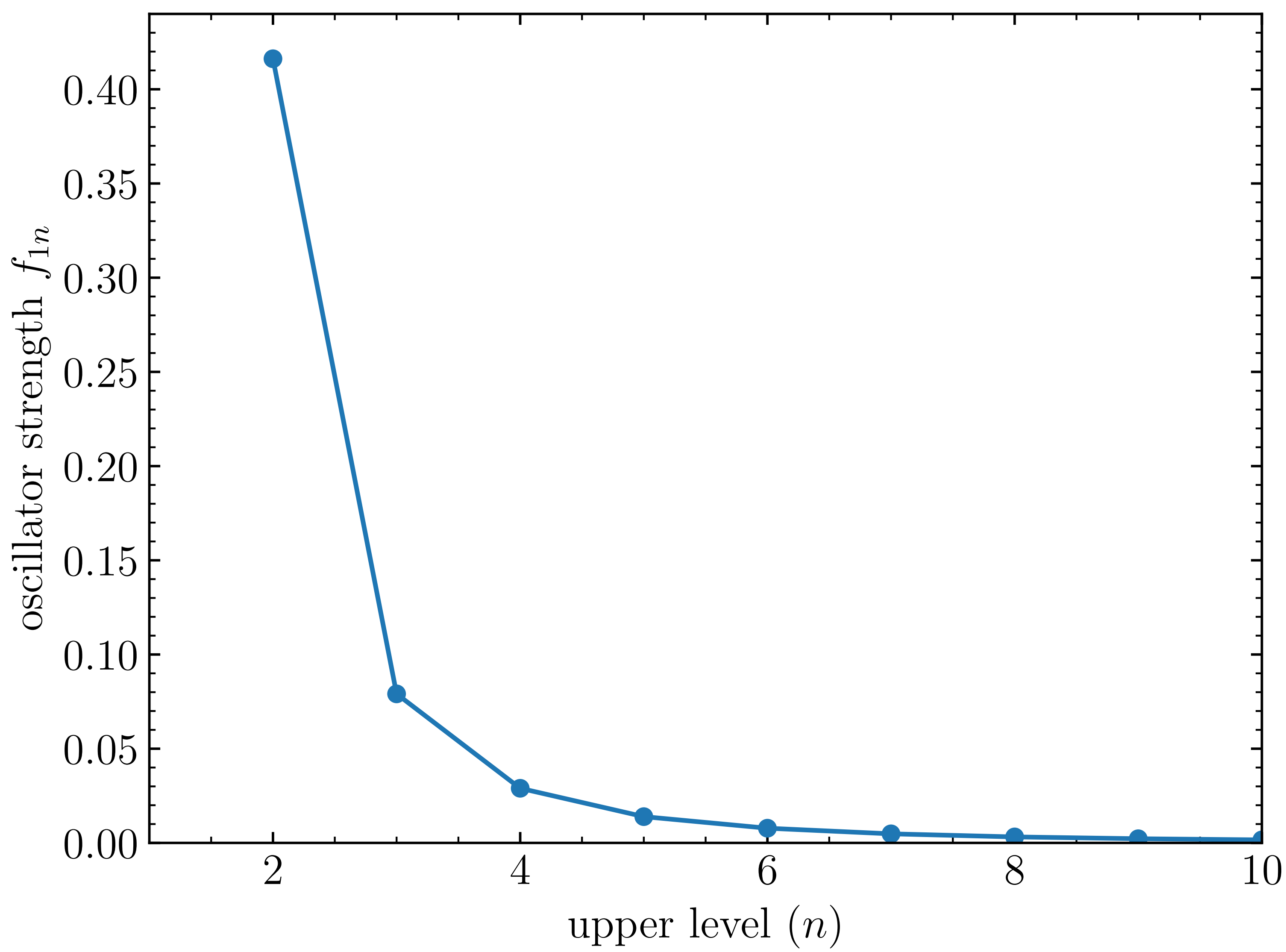
$$f_{1n} = \left(\frac{2^8}{3n^3}\right)\frac{(1-1/n)^{2n-4}}{(1+1/n)^{2n+4}}$$

Table 5.8 (Sun Kwok)

Oscillator strengths for some of the lower transitions of H

n_1, ℓ_1	n_2, ℓ_2	f	n_1, ℓ_1	n_2, ℓ_2	f
1,0	2,1 (Lyα)	0.4162	2,1	3,0 (Hα)	0.01359
	3,1 (Lyβ)	0.07910		3,2 (Hα)	0.6958
	4,1 (Lyγ)	0.02899		4,0 (Hβ)	0.003045
	5,1 (Lyδ)	0.01394		4,2 (Hβ)	0.1218
2,0	3,1 (Hα)	0.4349		5,0(Hγ)	0.001213
	4,1 (Hβ)	0.1028		5,2 (Hγ)	0.04437

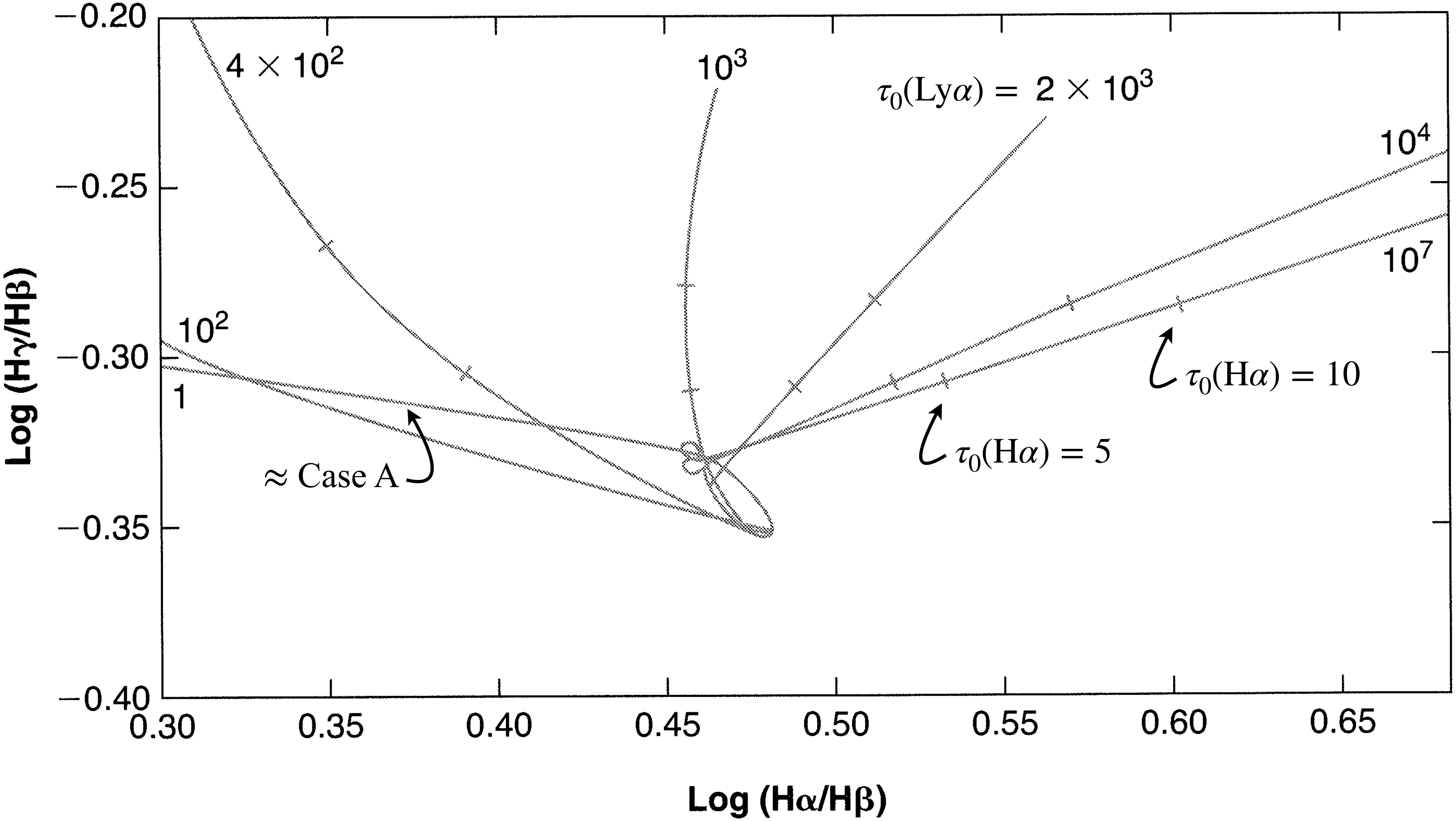
Oscillator Strengths of Lyman lines



absorption cross section:

$$\sigma_0 = f_{ji}\frac{\pi e^2}{m_e c}$$

- **Calculation of the emergent Balmer-line spectrum**
  - S_n = the number of Hn photons generated in the nebula per unit time by recombination and subsequent cascading.
  - K_n = total number of Hn photons emitted in the nebula per unit time = sum of contributions from recombination and from resonance fluorescence due to Lyman-line photons,
  - $K_n = S_n + \sum_{m=n}^{\infty} A_m P_m(\mathrm{H}n)$ if there is no absorption of the Balmer-line photons
  - Since S_n and $P_m(\mathrm{H}n)$ are known from the recombination theory and the A_m is known from the Lyman-line solution, the K_n can be calculated, giving the emergent Balmer-line spectrum.
  - Note that R_n, S_n, J_n, K_n, and A_n are proportional to the total number of photons; the equations are linear in these quantities; and the entire calculation can therefore be normalized to any S_n, for instance, S_4, the number of Hβ photons if there were no absorption effects.
  - The results for Hα/Hβ and Hβ/Hγ intensity ratios are shown in Figure as a function of $\tau_0(\mathrm{Ly}\alpha)$.

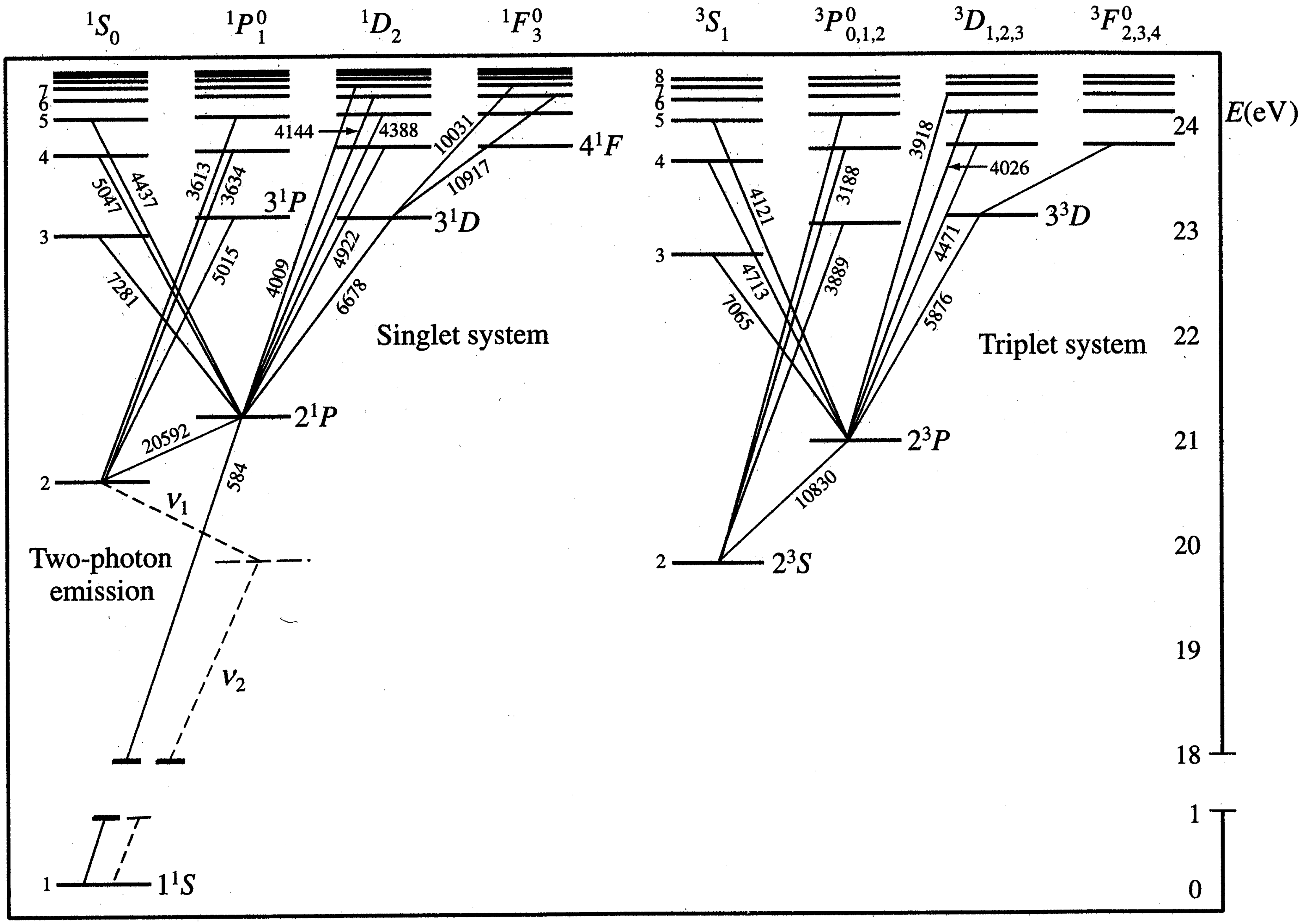


[Figure 4.3] Radiative transfer effects caused by finite optical depths in Lyman and Balmer lines. Ratios of the emitted fluxes are shown for homogeneous static isothermal nebulae at $T = 10^4$ K. The figure demonstrates the transition from Case A ($\tau_0(\mathrm{Ly}\alpha) \to 0$) to Case B ($\tau_0(\mathrm{Ly}\alpha) \to \infty$).

- In most nebulae, the optical depths in the Balmer lines are small. However, there could be situations in which the density $n(\mathrm{H}^0, 2\,^2S)$ is sufficiently high that some self-absorption does occur.
- The RT problem of the Balmer lines is a function of $\tau_0(\mathrm{Ly}\alpha)$, giving the optical radius in the Lyman lines, and $\tau_0(\mathrm{H}\alpha)$, giving the optical radius in the Balmer lines.
  - The equations are much more complicated, since now Balmer-line photons may be scattered or converted into Lyman-line photons and vice versa.
  - The same general type of formulation for the Lyman-line absorption can still be used.
- Figure 4.3 demonstrates the RT effect of Hα.
  - For $\tau_0(\mathrm{H}\alpha) = 0$, the effect of increasing $\tau_0(\mathrm{Ly}\alpha)$ is that Lyβ is converted into Hα + two-photon continuum. This increases the Hα/Hβ ratio.==> move to the right
  - For slightly larger $\tau_0(\mathrm{Ly}\alpha)$, Ly$\gamma$ photons are converted into Paα, Hα, Hβ, Lyα, and two-photon continuum photons. The main effect is to increase the strength of Hβ. ==> move downward and to the left.
  - For still larger $\tau_0(\mathrm{Ly}\alpha)$, as still higher Ly$n$ photons are converted. Hγ is also strengthened. ==> small loop as the conditions change from Case A to Case B.
  - For large $\tau_0(\mathrm{Ly}\alpha)$, as $\tau_0(\mathrm{H}\alpha)$, Hα is merely scattered (because any Lyβ photons it forms are quickly absorbed and converted back to Hα), and Hβ is absorbed and converted to Hα + Paα. This increases Hα/Hβ and Hγ/Hβ.

Helium Energy Levels

- Helium (Grotrian diagram)



The states can be divided into two separate groups because of the selection rule $\Delta S = 0$.

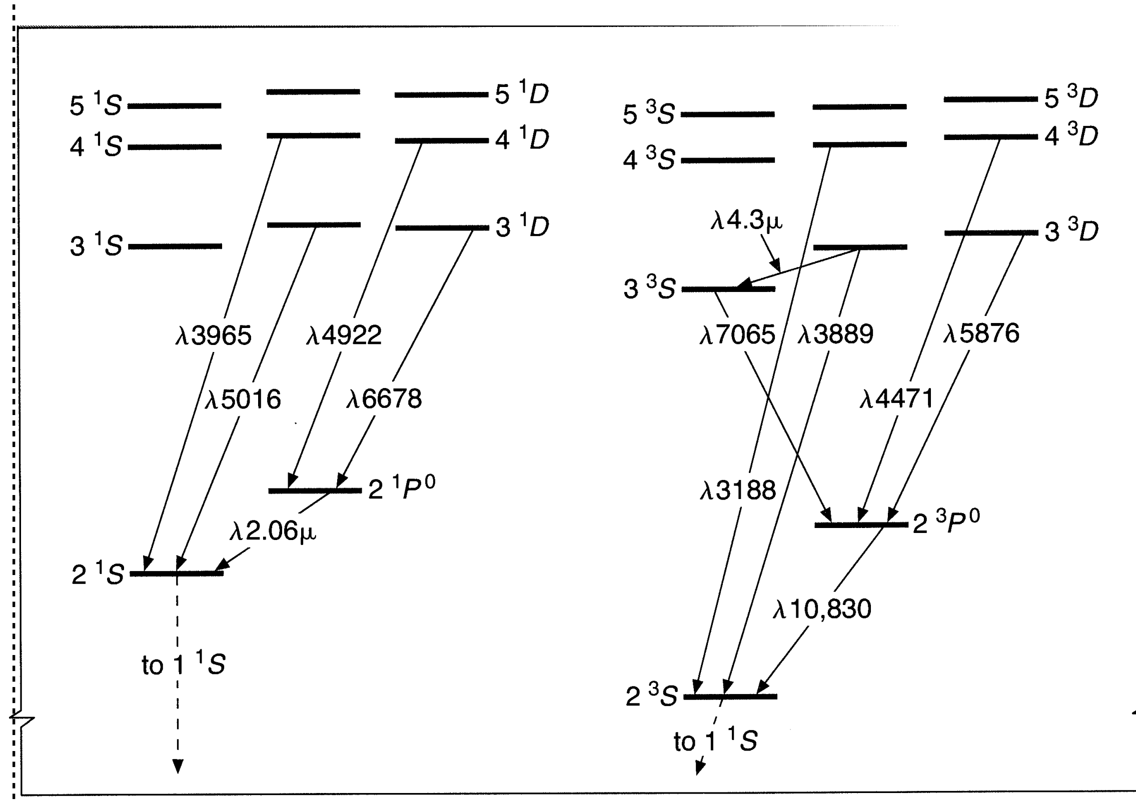
4.6 Radiative Transfer Effects in He I

- He I singlets:
  - The recombination radiation of He I singlets is very similar to that of H I. Case B is a good approximation for the He I Lyman lines.
- He I triplets:
  - Recombination to triplets tend to cascade down to $2\,^3S$.
  - The **He^0 $2\,^3S$ term is considerably more metastable than H^0 $2\,^2S$.** Thus, the number density $n(2\,^3S)$ is large and self-absorption effects are quite important.
  - Depopulation occurs (1) by photoionization, especially by H I Lyα, (2) by collisional transitions (excitations) to $2\,^1S$ and $2\,^1P^o$, or (3) by the strongly forbidden $2\,^3S - 1\,^1S$ radiative transition.

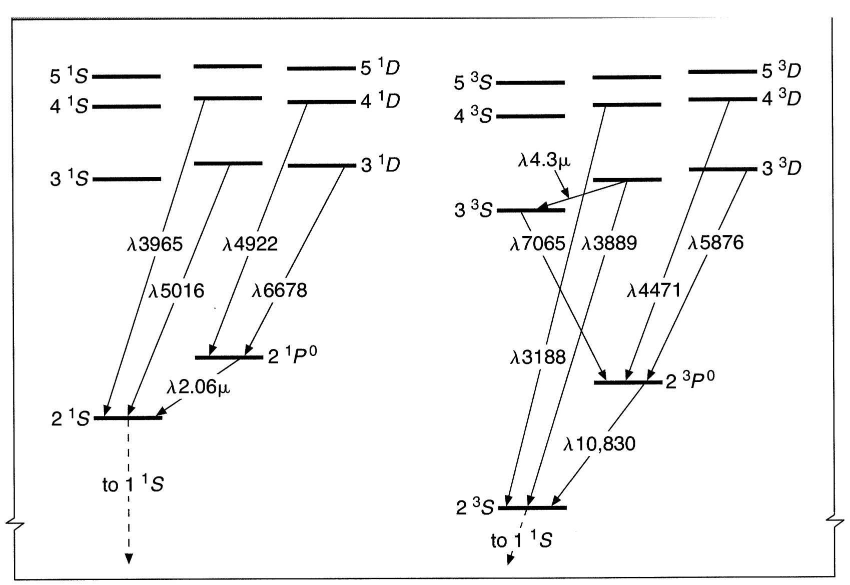
- $\lambda10830$ ($2\,^3S - 2\,^3P^o$) photons are scattered.
- $\lambda3889$ ($2\,^3S - 3\,^3P^o$) photons can be either (1) scattered or (2) converted to three lines $\lambda4.3\,\mu$m ($3\,^3S - 3\,^3P^o$) + $\lambda7065$ ($2\,^3P^o - 3\,^3S$) + $\lambda10830$ ($2\,^3S - 2\,^3P^o$) by resonance fluorescence.
- The probability of this conversion of $\lambda3889$ ($2\,^3S - 3\,^3P^o$) is

$$\frac{A_{3\,^3S,\,3\,^3P^o}}{A_{3\,^3S,\,3\,^3P^o} + A_{2\,^3S,\,3\,^3P^o}} \approx 0.10.$$

- At larger $\tau_0(\lambda10830)$, still higher members of the $2\,^3S - n\,^3P^o$ series are converted into longer wavelength photons.



- The RT problem is very similar to that for the Lyman lines.



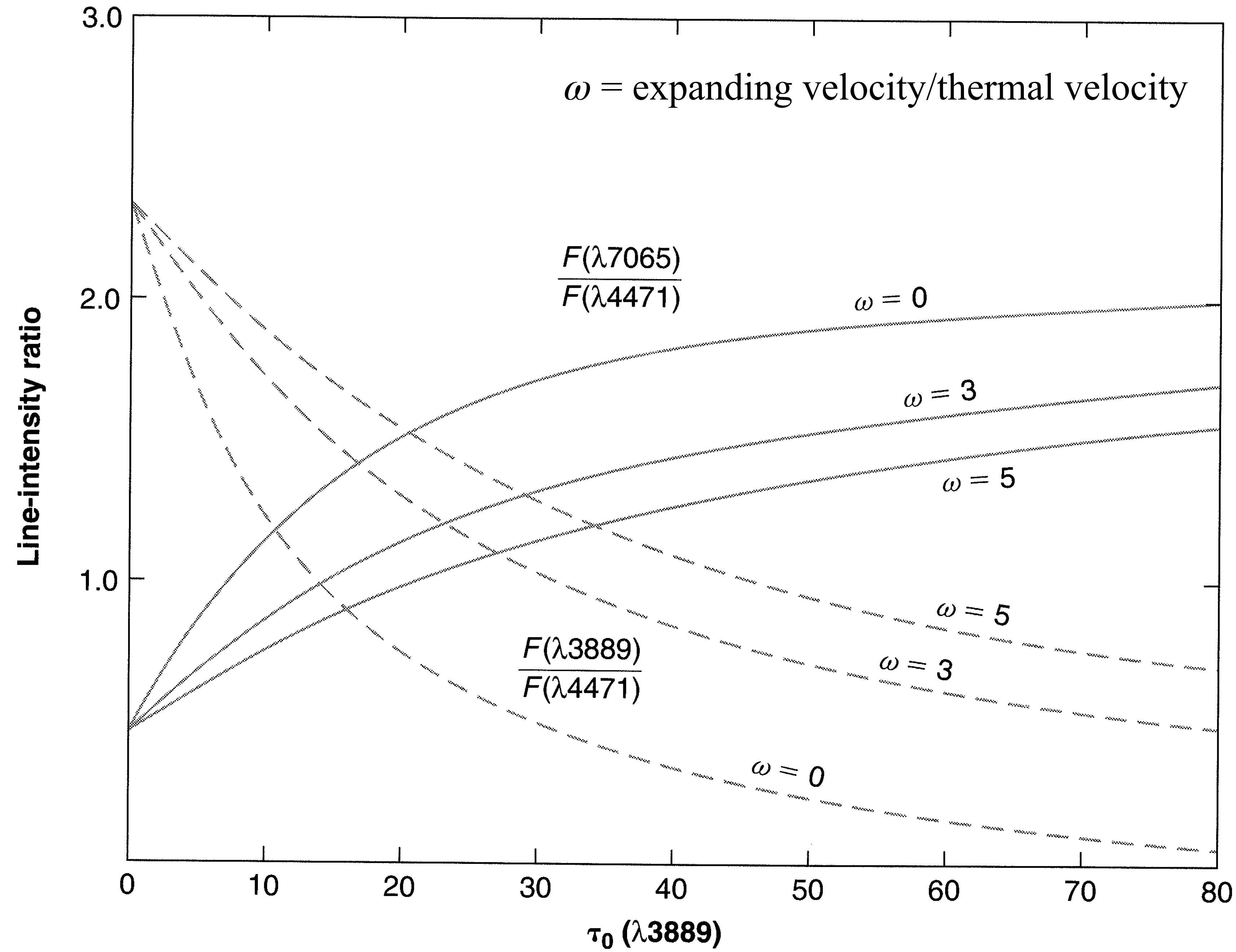


Figure 4.5

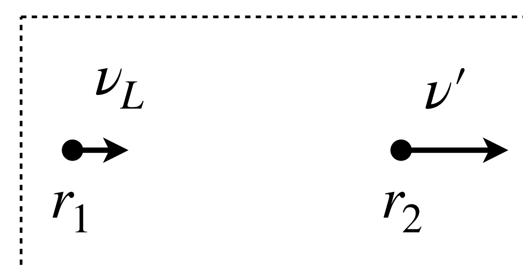
Radiative transfer effects due to finite optical depths in He I $\lambda3889$ $2\ {}^3S$–$3\ {}^3P^o$. Ratios of emergent fluxes of $\lambda7065$ and $\lambda3889$ to the flux in $\lambda4471$ are as a function of optical radius $\tau_0(\lambda3889)$ of homogeneous static ($\omega = 0$) and expanding ($\omega \neq 0$) isothermal nebulae at $T = 10{,}000$ K.

- Line broadening
  - The thermal Doppler widths of He I lines are smaller than those of H I lines, because of the larger mass of He. Therefore, turbulent or expansion velocity is relatively more important in broadening the He I lines.
- Consider a model spherical nebula expanding with a linear velocity of expansion (Hubble-like expansion).
  - $U_{\rm exp}(r) = \omega r \quad (0 \le r \le R)$, where ω is the constant, radial velocity gradient.
  - Photons emitted at r_1 will have a line profile centered at the line frequency ν_L in the local rest frame. They will encounter at r_2 ($> r_1$) material absorbing with a profile centered on the frequency

$$\nu'(r_1, r_2) = \nu_L \left(1 + \frac{\omega s}{c}\right),$$ where s is the distance between the points.

  - The optical depth from r_1 to the boundary of the nebula for a photon emitted at r_1 with frequency ν is

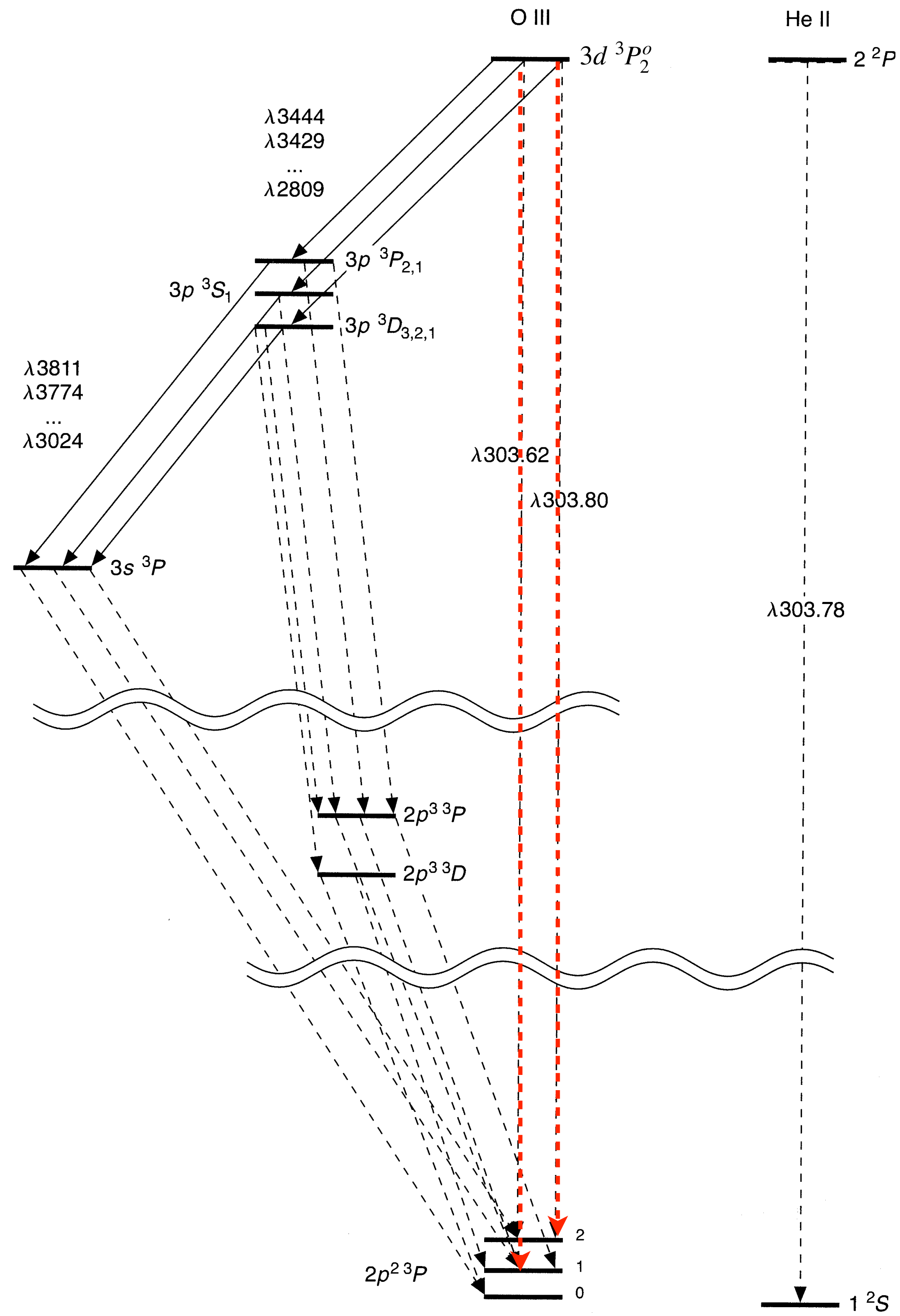
$$\tau_\nu = \int_0^{r_2=R} n(2\,^3S) k_{0l} \exp\left\{-\left[\frac{\nu - \nu'(r_1, r_2)}{\Delta\nu_D}\right]^2\right\} ds.$$



  - Increasing velocity of expansion tends to decrease the optical depth to the boundary, and thus to decrease the self-absorption effects. See Figure 4.5 to see the expansion velocity effect.

4.7 The Bowen Resonance-Fluorescence Mechanisms for O III and O I

- He II Lyα and O III
  - There is an accidental coincidence between the wavelengths of He II Lyα $\lambda 303.78$ and O III $\lambda 303.80$ ($2p^2\,{}^3P_2 - 3d\,{}^3P_2^o$)
- In the He^{++} zone,
  - there is some residual He$^+$, so He II Lyα emitted by recombination are scattered many times before they escape.
  - Consequently, there is a high density of He II Lyα photons.
  - Since O^{++} is also present in this zone, some of the He II Lyα photons are absorbed by it and excited the $3d\,{}^3P_2^o$ level of O III.
  - This level quickly decays through a radiative transition, to (1) $2p^2\,{}^3P_2$ with a probability $p = 0.74$ and $\lambda = 303.80$Å, (2) $2p^2\,{}^3P_1$ with $p = 0.24$ and $\lambda = 303.62$Å, and (3) $3p\,{}^3L_J$ with $p = 0.02$ and six wavelengths. The third route decays to lower levels.
  - These lines are observed in many planetary nebulae.

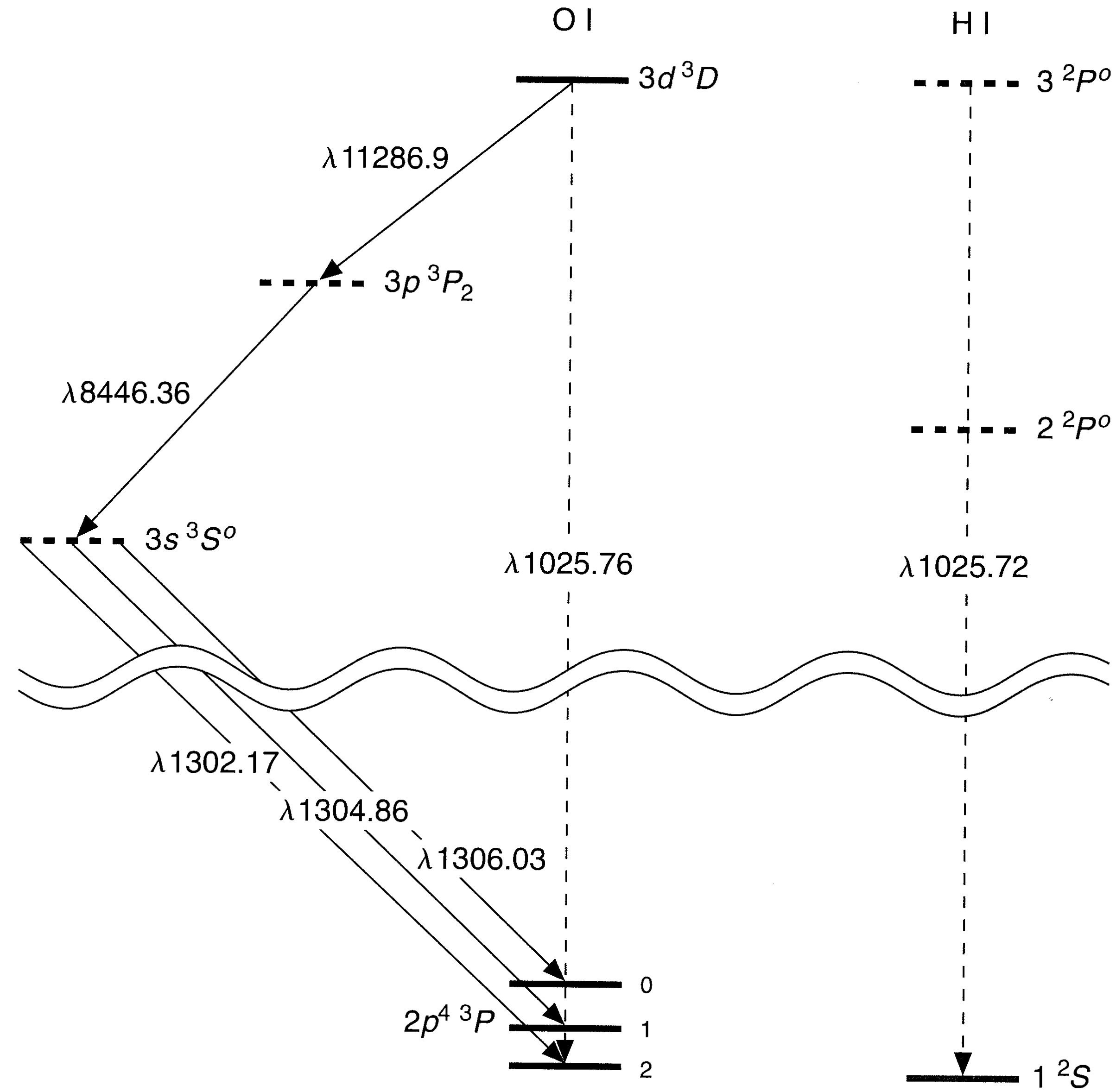


- H I and O I
  - A second accidental near-coincidence occurs between H I Lyβ $\lambda 1025.72$ and O I $\lambda 1025.76$ $(2p^{4}\,{}^{3}P_{2} - 2p^{3}3d\,{}^{3}D^{o}_{3})$
- In the H^{+} zone,
  - Some atomic oxygen exists, due to rapid charge exchange between O and H.

$$H + O^{+} \leftrightarrow H^{+} + O$$

H ionization energy = 13.6 eV
O ionization energy = 13.62 eV

  - Excitation of $2p^{3}3d\,{}^{3}D^{o}_{3}$ are followed by successive decays, producing
  - $\lambda 11286.9$ $(2p^{3}3p\,{}^{3}P_{2} - 2p^{3}3d\,{}^{3}D^{o}_{3})$, $\lambda 8446.36$ $(2p^{3}s\,{}^{3}S^{o}_{1} - 2p^{3}3p\,{}^{3}P_{2})$, $\lambda 1302.17,\ 1304.86, 1306.03$ $(2p^{4}\,{}^{3}P_{2,1,0} - 2p^{3}3s\,{}^{3}S^{o}_{1})$
  - The ratio of the transition probabilities of the last three multiplet is 3.4:2.0:0.7.



- Coincidence of O III resonance line $\lambda 374.432$ with N III two resonance lines $\lambda 374.434$ and $\lambda 374.442$

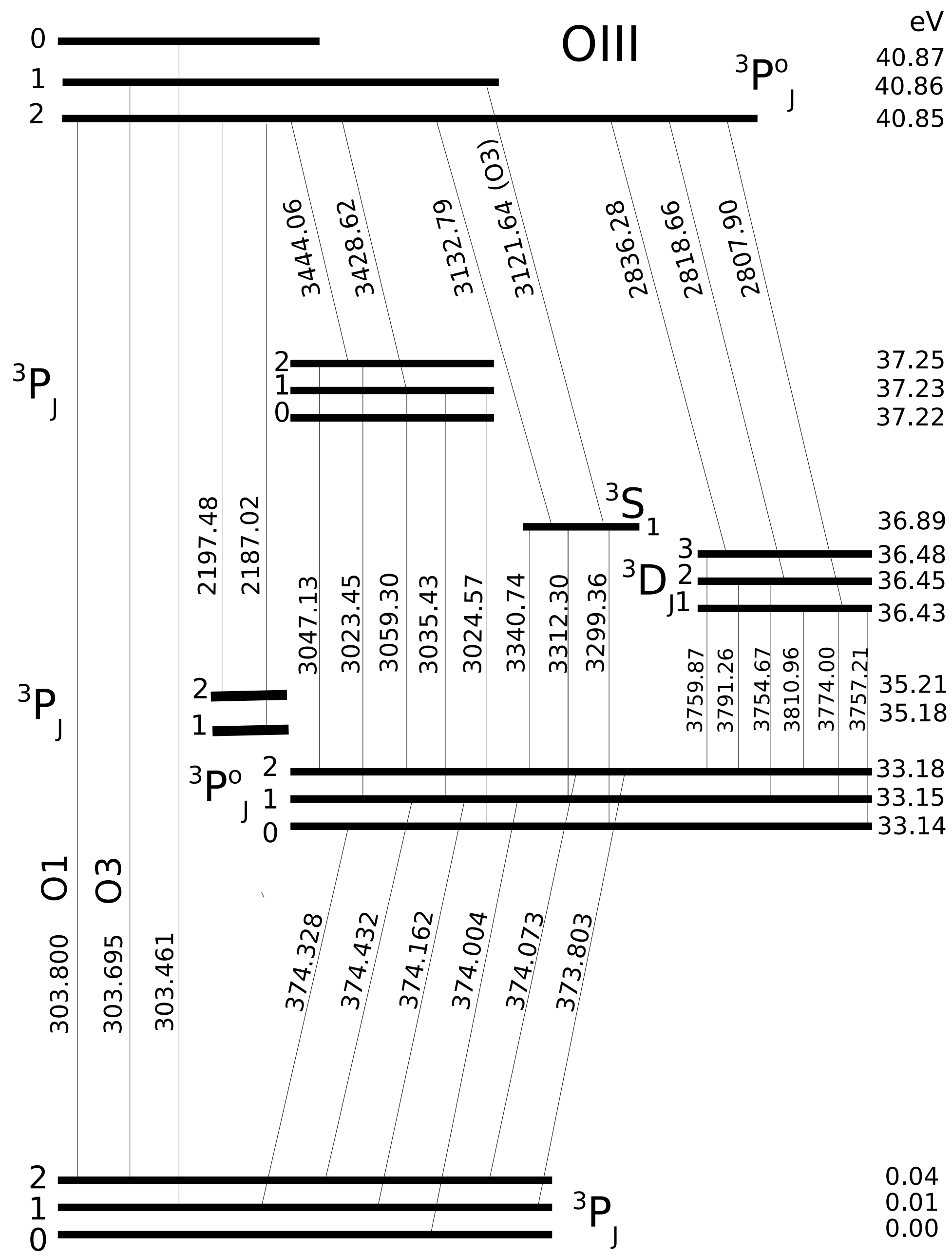


Fig. 6. A partial Grotrian diagram of O III that includes the most relevant O1 transitions and the λ 3121.64 line belonging to the O3 process. The level's configuration can be read from Table 3. The y-axis is not scaled linearly.

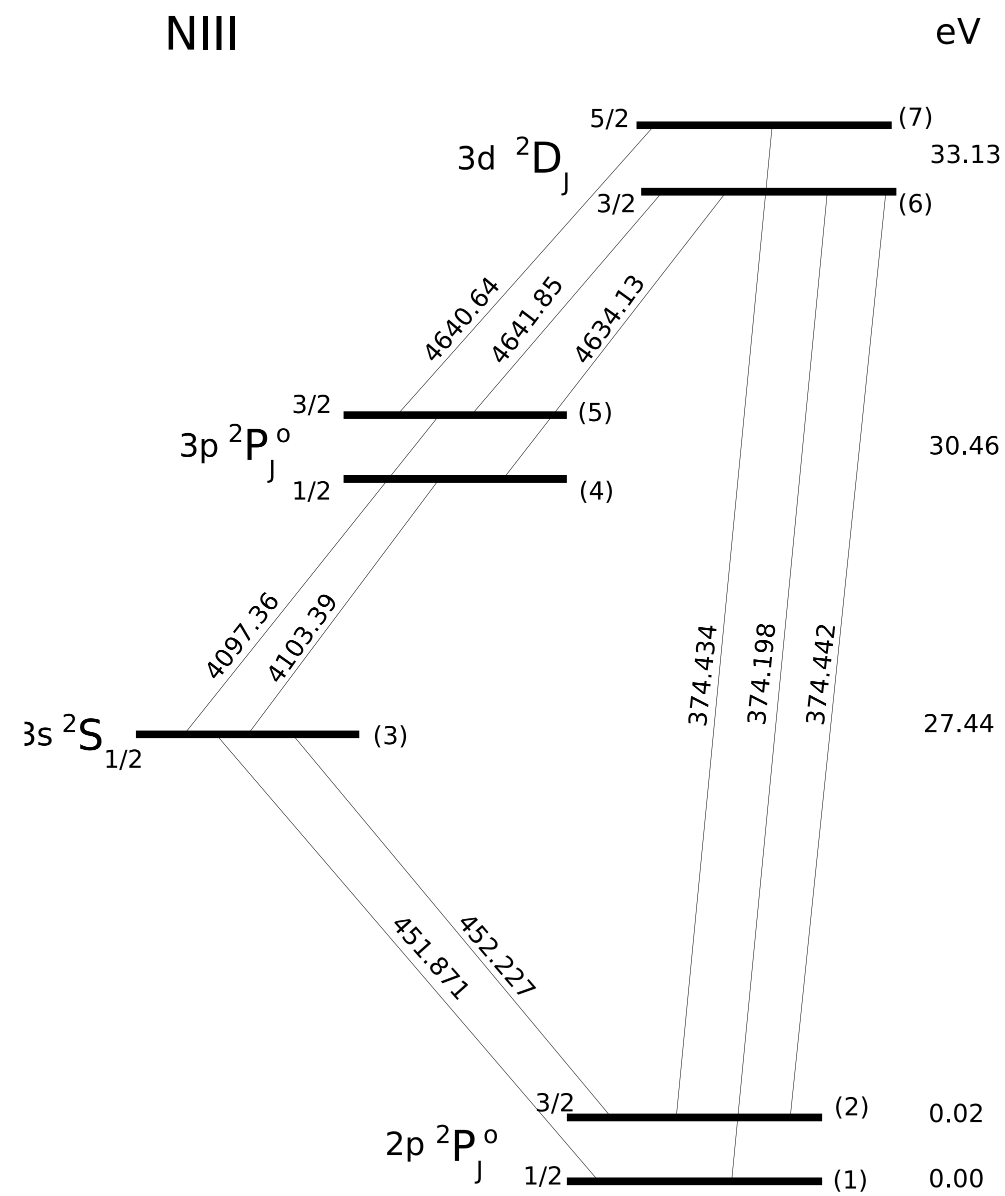


Fig. 13. A partial Grotrian diagram for N III. Levels are identified by a number index 1–7 to facilitate reading the text.

Selvelli et al. (2007, A&A, 464, 715)

4.8 Collisional Excitation in He I

- **Collisional excitation of H is negligible** in comparison with recombination in populating the excited levels in H II regions and planetary nebulae.
  - This is because the threshold for even the lowest level ($n = 2$) at 10.2 eV is large in comparison with the thermal energies at typical nebular temperatures.
- In He^0, the $2\,^3S$ level is highly metastable, and collisional excitation from it can be important.
  - In particular, collisional excitation to $2\,^3P^o$ is important and leads to emission of He I λ10830 Å.
  - In a sufficiently dense ($n_e \gg n_c$) nebula, the main mechanism for depopulating $2\,^3P^o$ is collisional transitions to $2\,^1S$ and $2\,^1P$.
  - The equilibrium population in $2\,^3S$ is given by the balance between recombination to all triplet levels (which eventually cascade down to $2\,^3S$), and collisional depopulation of $2\,^3S$.

$$n_e n(\mathrm{He}^+)\alpha_\mathrm{B}(\mathrm{He}^0, n\,^3L) = n_e n(2\,^3S)\left[q_{2\,^3S,\,1\,^1S} + q_{2\,^3S,\,2\,^1P^o}\right]$$

  - The rate of collisional population of $2\,^3P^o$ is then

$$n_e n(2\,^3S) q_{2\,^3S,\,2\,^3P^o} = \frac{n_e n(\mathrm{He}^+) q_{2\,^3S,\,2\,^3P^o}}{\left[q_{2\,^3S,\,1\,^1S} + q_{2\,^3S,\,2\,^1P^o}\right]}\alpha_\mathrm{B}(\mathrm{He}^0, n\,^3L)$$

  - The recombination rate is $n_e n(\mathrm{He}^+)\alpha^{\mathrm{eff}}_{\lambda 10830}$. At a typical temperature $T = 10^4$ K, the ratio of collisional excitation to recombination is about 8. In other words, collisional excitation from $2\,^3S$ completely dominates the emission of λ10830.

- Although the collisional transition rates from $2\,^3S$ to $2\,^1S$ and $2\,^1P^o$ are smaller than to $2\,^3P^o$, the recombination rates of population of these singlet levels are also smaller, and the collisions are also important in the population of $2\,^1S$ and $2\,^1P^o$.