Interstellar Medium (ISM)

Week 4

2025 March 24 (Monday), 9AM

updated 03/05, 21:38

선광일 (Kwangil Seon)
KASI / UST

Neutral Medium

- Intrinsic Line Profile
- Excitation & De-excitation
- Optical/UV Absorption Line
- Curve of Growth
- H I 21 cm line
- Cold Neutral Medium & Warm Neutral Medium

Line Profile: Classical model

- ***Lorentz Oscillator Model*** **to describe the interaction between atoms and electric fields**
  - The electron (with a small mass) is bound to the nucleus of the atom (with a much larger mass) by a force that behaves according to Hooke's Law (a spring-like force).
  - An applied electric field would then interact with the charge of the electron, causing "stretching" or "compression" of the spring.
  - ***The electron's equation of motion:***

$$\boxed{m\ddot{\mathbf{x}} = -k\mathbf{x} + \mathbf{F}_{\rm ext} + \mathbf{F}_{\rm rad}}$$

Here, $m =$ electron mass

$$k = m\omega_0^2\ , \ \ \text{where } k = \text{spring constant}$$

$$\omega_0 = \text{natural (fundamental or resonant) freqency}$$

$\mathbf{F}_{\rm ext} =$ external force, driving force, or external electric field

$\mathbf{F}_{\rm rad} =$ radiation reaction force (radiation damping)
the damping of a charge's motion which arises because of the emission of radiation

[1] Spontaneous Emission : Damping, Free Oscillator

- **Undriven Harmonically Bound Particles** (free oscillator)
  - Since an oscillating electron represents a continuously accelerating charge, the electron will radiate energy.
  - The energy radiated away must come from the particle's own energy (energy conservation). In other words, **there must be a force acting on a particle by virtue of the radiation it produces. This is called the *radiation reaction force*.**
  - Let's derive the formula for the radiation reaction force from the fact that the energy radiated must be compensated for by the work done against the radiation reaction force.
  - On one hand, the radiative loss rate of energy, averaged over one cycle of the oscillating dipole, can be represented by the radiative reaction force:

$$\frac{dW}{dt} = \langle \mathbf{F}_{\rm rad} \cdot \dot{\mathbf{x}} \rangle$$

  - On the other hand, from the *Larmor's formula* for a dipole, the radiative loss will be:

$$\frac{dW}{dt} = -\frac{2e^2 \left\langle |\ddot{\mathbf{x}}|^2 \right\rangle}{3c^3}$$

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3} e^2 |\bar{x}(\omega)|^2$$

$$\text{angular frequency} : \omega = 2\pi\nu$$

[1] Spontaneous Emission : Abraham-Lorentz formula

$$\therefore \quad \langle \mathbf{F}_{\mathrm{rad}} \cdot \dot{\mathbf{x}} \rangle = -\frac{2e^2 \left\langle |\ddot{\mathbf{x}}|^2 \right\rangle}{3c^3}$$

Here, $\left\langle |\ddot{\mathbf{x}}|^2 \right\rangle \equiv \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} \ddot{\mathbf{x}} \cdot \ddot{\mathbf{x}} dt$ where τ is the oscillation period.

$$= \frac{1}{\tau}\ddot{\mathbf{x}} \cdot \dot{\mathbf{x}}|_{-\tau/2}^{\tau/2} - \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} \dddot{\mathbf{x}} \cdot \dot{\mathbf{x}} dt$$

We assume that the initial and final states are the same: $\ddot{\mathbf{x}} \cdot \dot{\mathbf{x}}(-\tau/2) = \ddot{\mathbf{x}} \cdot \dot{\mathbf{x}}(\tau/2)$

Then,

$$\left\langle |\ddot{\mathbf{x}}|^2 \right\rangle = -\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} \dddot{\mathbf{x}} \cdot \dot{\mathbf{x}} dt = -\langle \dddot{\mathbf{x}} \cdot \dot{\mathbf{x}} \rangle \quad \rightarrow \quad \langle \mathbf{F}_{\mathrm{rad}} \cdot \dot{\mathbf{x}} \rangle = \frac{2e^2 \langle \dddot{\mathbf{x}} \cdot \dot{\mathbf{x}} \rangle}{3c^3}$$

Therefore, we can obtain $\mathbf{F}_{\mathrm{rad}} = \frac{2e^2 \dddot{\mathbf{x}}}{3c^3}$: Abraham-Lorentz formula

- **Abraham-Lorentz formula**:

$$\mathbf{F}_{\text{rad}} = \frac{2e^2\dddot{\mathbf{x}}}{3c^3}$$

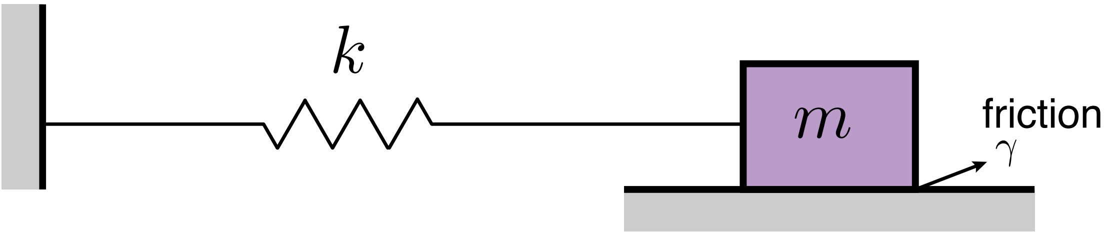
  - This formula depends on the derivative of acceleration. This increases the degree of the equation of motion of a particle and can lead to some nonphysical behavior if not used properly and consistently.
  - For a simple harmonic oscillator with a frequency ω_0 , we can avoid the difficulty by using

$$\dddot{\mathbf{x}} = -\omega_0^2\dot{\mathbf{x}} \quad \longleftarrow \quad \begin{aligned} x(t) &= x_0\cos(\omega_0 t) \qquad & \ddot{x}(t) &= -\omega_0^2 x(t) \\ \dot{x}(t) &= -\omega_0 x_0 \sin(\omega_0 t) \qquad & \dddot{x}(t) &= -\omega_0^2 \dot{x}(t) \end{aligned}$$

  - ***This is a good assumption as long as the energy is to be radiated on a time scale that is long compared to the period of oscillation*** **(**$\gamma \ll \omega_0$**).** In this regime, ***radiation reaction may be considered as a perturbation on the particle's motion***. We then rewrite the radiation reaction force as

$$\boxed{\mathbf{F}_{\text{rad}} = -\frac{2e^2\omega_0^2}{3c^3}\dot{\mathbf{x}} = -m\gamma\dot{\mathbf{x}},} \qquad \gamma \equiv \frac{2e^2\omega_0^2}{3mc^3}$$

: damping constant



$$\boxed{m\ddot{\mathbf{x}} + k\mathbf{x} + m\gamma\dot{\mathbf{x}} = 0}$$

This is the equation for a string-mass system subject to friction damping.

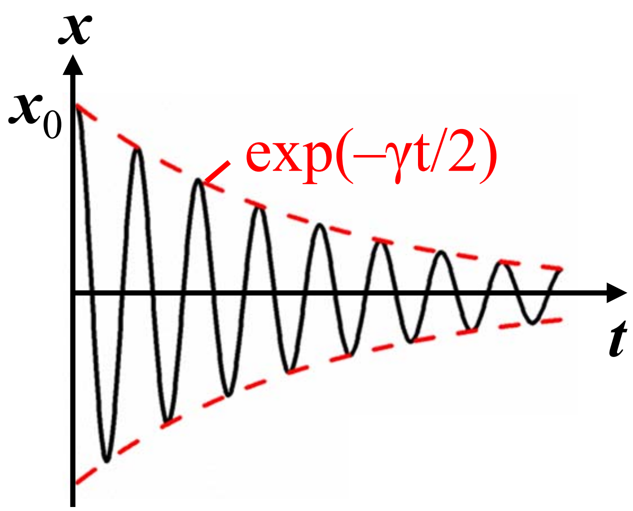
- Therefore, the equation of motion of the electron in a Lorentz atom is

$$\ddot{\mathbf{x}} + \gamma\dot{\mathbf{x}} + \omega_0^2\mathbf{x} = 0$$

  - This equation may be solved by assuming that $x(t) \propto e^{\alpha t}$.

$$\alpha^2 + \gamma\alpha + \omega_0^2 = 0 \;\rightarrow\; \alpha = -(\gamma/2) \pm \sqrt{(\gamma/2)^2 - \omega_0^2}$$
$$= -\gamma/2 \pm i\omega_0 + \mathcal{O}(\gamma^2/\omega_0^2)$$

Here, we assumed $\gamma \ll \omega_0$.



  - Assuming initial conditions: $x(0) = x_0,\; \dot{x}(0) = 0 \text{ at } t = 0$
  - we have

$$x(t) = \frac{1}{2}x_0\left[e^{-(\gamma/2 - i\omega_0)t} + e^{-(\gamma/2 + i\omega_0)t}\right] = x_0 e^{-\gamma t/2}\cos\omega_0 t \longrightarrow$$

This is a damping oscillator.

- Power spectrum:

$$\bar{x}(\omega) = \frac{1}{2\pi}\int_0^\infty x(t)e^{i\omega t}dt = \frac{x_0}{4\pi}\left[\frac{1}{\gamma/2 - i(\omega+\omega_0)} + \frac{1}{\gamma/2 - i(\omega-\omega_0)}\right]$$

  - This becomes large in the vicinity of $\omega = \omega_0$ and $\omega = -\omega_0$.
  - We are ultimately interested only in positive frequencies, and only in regions in which the values become large. Therefore, we obtain

$$\bar{x}(\omega) \approx \frac{x_0}{4\pi}\frac{1}{\gamma/2 - i(\omega-\omega_0)}, \quad |\bar{x}(\omega)|^2 = \left(\frac{x_0}{4\pi}\right)^2\frac{1}{(\omega-\omega_0)^2 + (\gamma/2)^2}$$

- Spontaneous Emission: Line profile

- Recall the Larmor's formula: $$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3} e^2 |\bar{x}(\omega)|^2$$

- Energy radiated per unit frequency:

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3} \frac{e^2 x_0^2}{(4\pi)^2} \frac{1}{(\omega-\omega_0)^2 + (\gamma/2)^2} = \frac{1}{2} m \left(\frac{\omega^4}{\omega_0^2}\right) x_0^2 \frac{\gamma/2\pi}{(\omega-\omega_0)^2 + (\gamma/2)^2}$$

$$\approx \frac{1}{2} m\omega_0^2 x_0^2 \frac{\gamma/2\pi}{(\omega-\omega_0)^2 + (\gamma/2)^2}$$

- For a harmonic oscillator, note that the equation of motion is $\mathbf{F} = -k\mathbf{x} = -m\omega_0^2\mathbf{x}$, spring constant is $k = m\omega_0^2$, and the potential energy (energy stored in spring) is $(1/2)kx_0^2$.

- From $$\int_{-\infty}^{\infty} \frac{\gamma/2\pi}{(\omega-\omega_0)^2 + (\gamma/2)^2} d\omega = \frac{1}{\pi} \tan^{-1}\{2(\omega-\omega_0)/\gamma\}|_{-\infty}^{\infty} = 1$$

- Note that the total emitted energy is equal to the initial potential energy of the oscillator:

$$W = \int_0^{\infty} \frac{dW}{d\omega} d\omega = \frac{1}{2} k x_0^2$$

- Profile of the emitted spectrum: $$\boxed{\phi(\omega) = \frac{\gamma/2\pi}{(\omega-\omega_0)^2 + (\gamma/2)^2}}$$

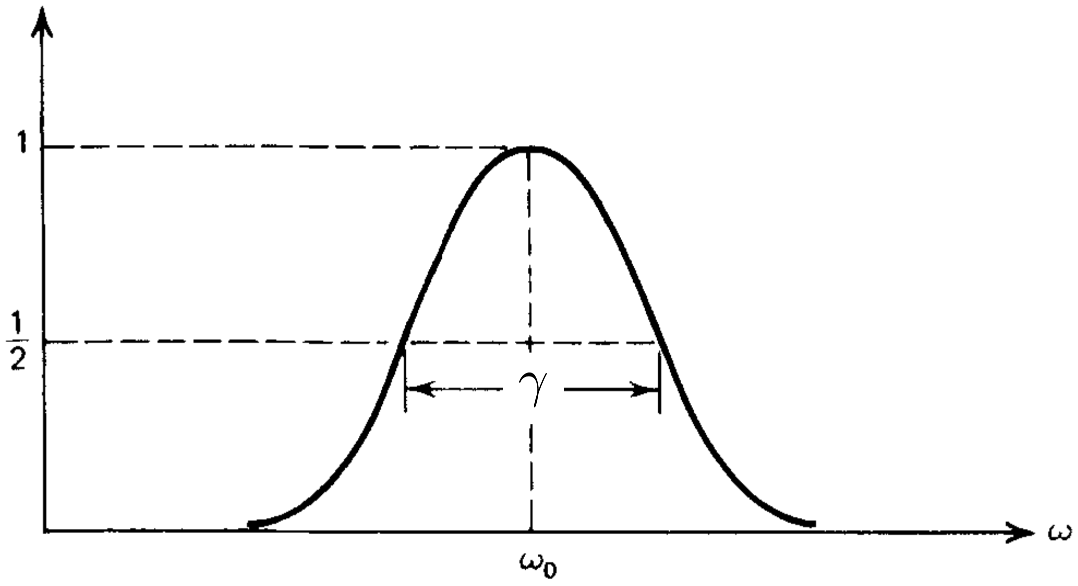
This is the Lorentz (natural) line profile.

- Damping constant is the full width at half maximum (FWHM).

$$\phi(\omega) = \frac{\gamma/2\pi}{(\omega-\omega_0)^2 + (\gamma/2)^2}$$

$$\boxed{\phi(\nu) = \frac{\gamma/4\pi^2}{(\nu-\nu_0)^2 + (\gamma/4\pi)^2}}$$

Note $\phi(\omega)d\omega = \phi(\nu)d\nu$



- **The line width $\Delta\omega = \gamma$ is a universal constant** when expressed in terms of wavelength:

$$\lambda = \frac{2\pi c}{\omega} \qquad (\omega = 2\pi\nu)$$

$$\Delta\lambda = 2\pi c\frac{\Delta\omega}{\omega^2} = 2\pi c\frac{2}{3}\frac{r_e}{c} \longleftarrow \left(\Delta\omega = \gamma = \frac{2}{3}r_e\frac{\omega_0^2}{c}\right)$$

$$= \frac{4}{3}\pi r_e$$

$$= 1.2 \times 10^{-4}\text{Å}$$

However, **in Quantum Mechanics, the line width is not a universal constant.**

[2] Absorption/Scattering : Driven Oscillator

- **Driven Harmonically Bound Particles** (forced oscillators)
  - Electron's equation of motion (electric charge = $-e$): $\mathbf{F}_{\text{ext}} = -e\mathbf{E}_0 e^{i\omega t}$

$$\ddot{\mathbf{x}} + \gamma\dot{\mathbf{x}} + \omega_0^2\mathbf{x} = -\frac{e\mathbf{E}_0}{m}e^{i\omega t}$$

  - A particular solution for this inhomogeneous differential equation:

$$\mathbf{x} = \mathbf{x}_0 e^{i\omega t} \equiv |\mathbf{x}_0| e^{i(\omega t+\delta)} \quad \rightarrow \quad (-\omega^2 + i\omega\gamma + \omega_0^2)\mathbf{x}_0 e^{i\omega t} = -\frac{e\mathbf{E}_0}{m}e^{i\omega t}$$

$$\mathbf{x}_0 = \frac{(e/m)\mathbf{E}_0}{(\omega^2-\omega_0^2) - i\omega\gamma}$$

$$\mathbf{x}_0 = |\mathbf{x}_0|e^{i\delta} \propto (\omega^2-\omega_0^2) + i\omega\gamma \quad \rightarrow \quad \delta = \tan^{-1}\left(\frac{\omega\gamma}{\omega^2-\omega_0^2}\right)$$

The response is slightly out of phase with respect to the imposed field.

  - Time-averaged total power radiated is given by

$$P = \left\langle \frac{dW}{dt} \right\rangle = \frac{2e^2\left\langle|\ddot{\mathbf{x}}|^2\right\rangle}{3c^3} = \frac{e^2\omega^4|\mathbf{x}_0|^2}{3c^3}$$

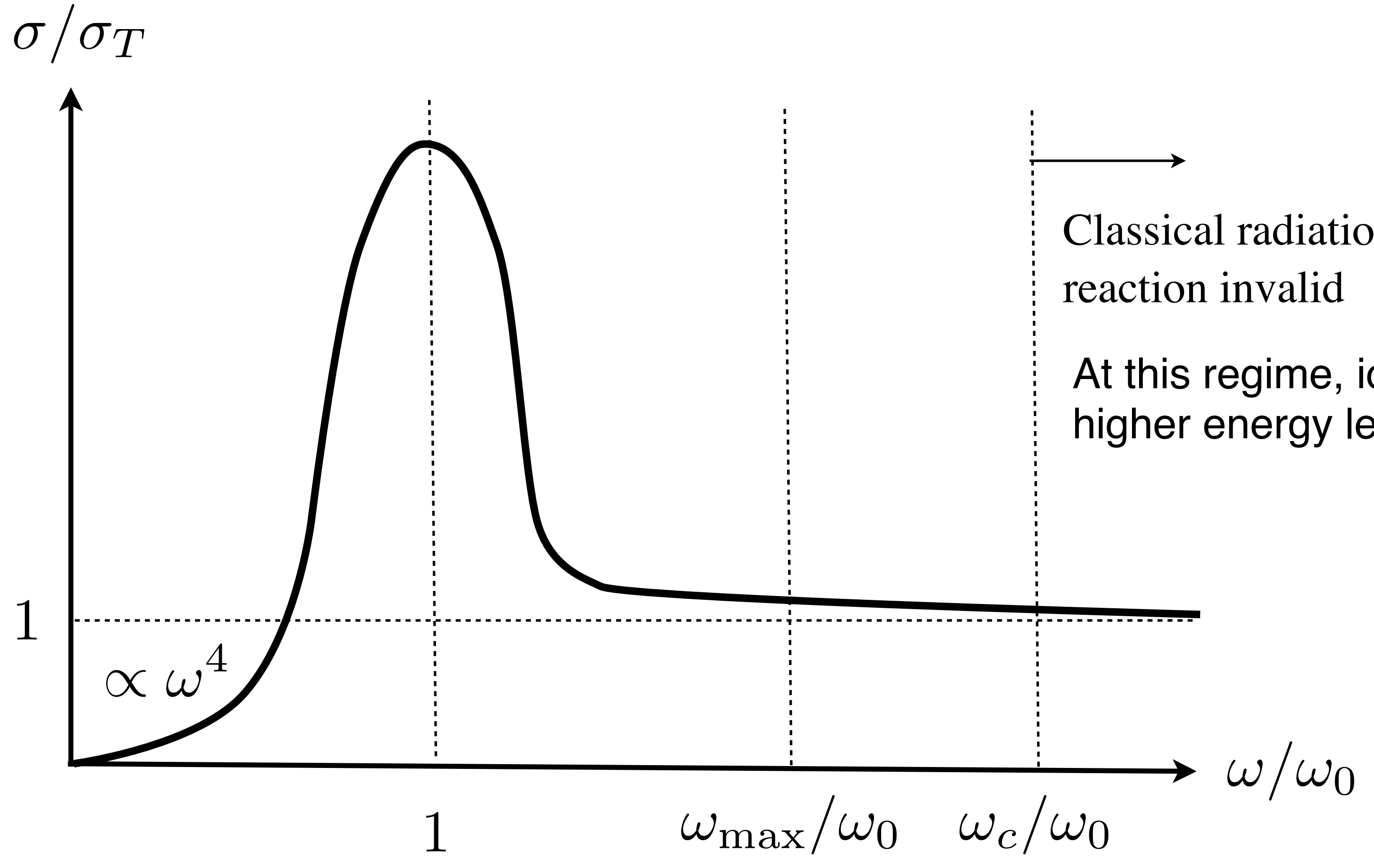
$$= \frac{e^4E_0^2}{3m^2c^3}\frac{\omega^4}{(\omega^2-\omega_0^2)^2 + (\omega\gamma)^2}$$

- Scattering cross section:

Poynting flux:

$$\sigma_{\rm sca} \equiv \frac{\langle P \rangle}{\langle S \rangle}, \quad \langle S \rangle = \frac{c}{8\pi} E_0^2 \quad \longrightarrow \quad \sigma_{\rm sca}(\omega) = \frac{8\pi e^4}{3m^2c^4} \frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + (\omega\gamma)^2}$$

$$= \sigma_T \frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + (\omega\gamma)^2}$$



Classical radiation
reaction invalid

At this regime, ionization and excitation to higher energy levels will occur.

- Limiting Cases of Interest

(a) $\omega \gg \omega_0$ (Thomson scattering by free electron)

$$\boxed{\sigma_{\text{sca}} = \sigma_T = \frac{8\pi}{3} r_e^2}$$

- At high incident energies, the binding becomes negligible. Therefore, this corresponds to the case of a free electron.

(b) $\omega \ll \omega_0$ (Rayleigh scattering by bound electron)

$$\boxed{\sigma_{\text{sca}} = \sigma_T \left(\frac{\omega}{\omega_0}\right)^4 = \sigma_T \left(\frac{\lambda_0}{\lambda}\right)^4}$$

- Rayleigh scattering refers to the ***scattering of light by particles smaller than the wavelength of the light.***
- The strong wavelength dependence of the scattering means that shorter (blue) wavelengths are scattered more strongly than longer wavelengths.
- (blue color of the sky) The dependence results in the indirect blue light coming from all regions of the sky.
- (red color of the sun at sunset) Conversely, glancing toward the Sun, the colors that were not scattered away - the longer wavelengths such as red and yellow light - are directly visible, giving the Sun itself a slightly yellowish color.
- However, viewed from space, the sky is black and the Sun is white.

- Absorption/Scattering : Line Profile

(c) $\omega \approx \omega_0$ (resonance scattering of line radiation)

$$\sigma_{\rm sca}(\omega) \approx \sigma_T \frac{\omega_0^4}{(\omega-\omega_0)^2(2\omega_0)^2 + (\omega_0\gamma)^2}$$

$$= \sigma_T \frac{\omega_0^2/4}{(\omega-\omega_0)^2 + (\gamma/2)^2}$$

$$\sigma_T \frac{\omega_0^2}{4} = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 \times \frac{1}{4} \times \left(\gamma \frac{3}{2}\frac{mc^3}{e^2\omega_0^2}\right) = 2\pi^2 \frac{e^2}{mc}(\gamma/2\pi)$$

Note that $\nu = \omega/2\pi$ and $\sigma_\nu = 2\pi\sigma_\omega$.

$$\sigma_\omega = \frac{2\pi^2 e^2}{m_e c}\frac{\gamma/2\pi}{(\omega-\omega_0)^2 + (\gamma/2)^2}$$

$$\sigma_\nu = \frac{\pi e^2}{m_e c}\frac{\gamma/4\pi^2}{(\nu-\nu_0)^2 + (\gamma/4\pi)^2}$$

- In the neighborhood of the resonance, ***the shape of the absorption/scattering cross-section is the same as the (spontaneous) emission line profile from the free oscillator.*** ***We already obtained the same conclusion, in the previous lecture.***
- Total scattering cross section is

$$\int_0^\infty \sigma_\nu d\nu = \frac{\pi e^2}{m_e c}$$

- ***Resonance line***
  - A spectral line caused by an electron jumping **between the ground state and the first energy level in an atom or ion**. It is the longest wavelength line produced by a jump to or from the ground state.
  - Because the majority of electrons are in the ground state in many astrophysical environments, and because the energy required to reach the first level is the least needed for any transition, resonance lines are the strongest lines in the spectrum for any given atom or ion.

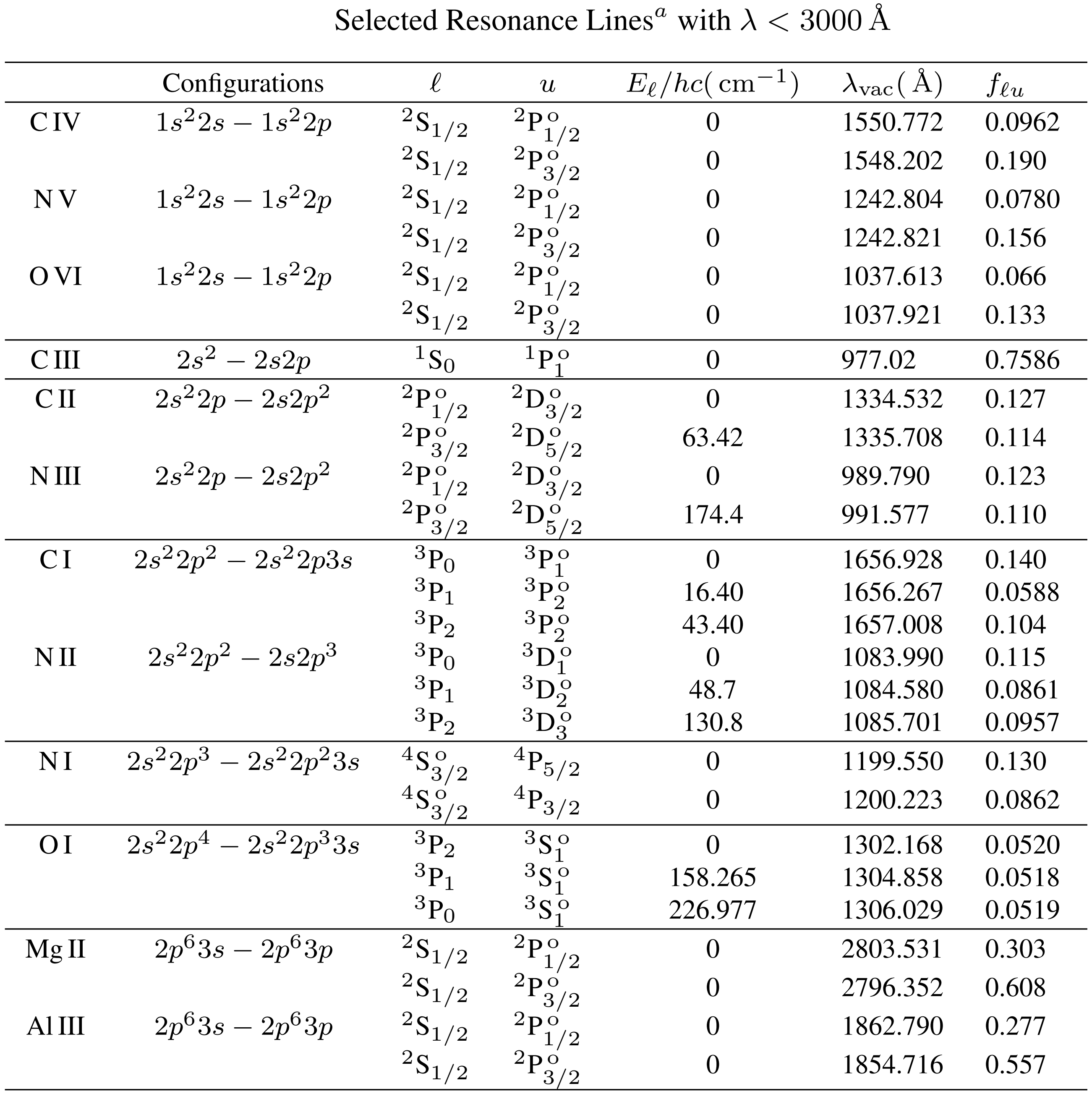
- ***In the quantum theory of spectral lines,***

  we obtain similar formulas, which are conveniently stated in terms of the classical results as

$$\sigma_\nu = f_{nn'} \frac{\pi e^2}{m_e c} \frac{\gamma / 4\pi^2}{(\nu - \nu_0)^2 + (\gamma / 4\pi)^2}$$

$$\int_0^\infty \sigma_\nu d\nu = f_{nn'} \frac{\pi e^2}{m_e c}$$

where $f_{nn'}$ is called the ***oscillator strength*** or ***f-value*** for the transition between states n and n'.



Selected Resonance Lines[a] with $\lambda < 3000$ Å

	Configurations	ℓ	u	$E_\ell/hc(\mathrm{cm}^{-1})$	$\lambda_{\mathrm{vac}}(\text{Å})$	$f_{\ell u}$
C IV	$1s^22s - 1s^22p$	$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^\mathrm{o}_{1/2}$	0	1550.772	0.0962
		$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^\mathrm{o}_{3/2}$	0	1548.202	0.190
N V	$1s^22s - 1s^22p$	$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^\mathrm{o}_{1/2}$	0	1242.804	0.0780
		$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^\mathrm{o}_{3/2}$	0	1242.821	0.156
O VI	$1s^22s - 1s^22p$	$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^\mathrm{o}_{1/2}$	0	1037.613	0.066
		$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^\mathrm{o}_{3/2}$	0	1037.921	0.133
C III	$2s^2 - 2s2p$	$^1\mathrm{S}_0$	$^1\mathrm{P}^\mathrm{o}_1$	0	977.02	0.7586
C II	$2s^22p - 2s2p^2$	$^2\mathrm{P}^\mathrm{o}_{1/2}$	$^2\mathrm{D}^\mathrm{o}_{3/2}$	0	1334.532	0.127
		$^2\mathrm{P}^\mathrm{o}_{3/2}$	$^2\mathrm{D}^\mathrm{o}_{5/2}$	63.42	1335.708	0.114
N III	$2s^22p - 2s2p^2$	$^2\mathrm{P}^\mathrm{o}_{1/2}$	$^2\mathrm{D}^\mathrm{o}_{3/2}$	0	989.790	0.123
		$^2\mathrm{P}^\mathrm{o}_{3/2}$	$^2\mathrm{D}^\mathrm{o}_{5/2}$	174.4	991.577	0.110
C I	$2s^22p^2 - 2s^22p3s$	$^3\mathrm{P}_0$	$^3\mathrm{P}^\mathrm{o}_1$	0	1656.928	0.140
		$^3\mathrm{P}_1$	$^3\mathrm{P}^\mathrm{o}_2$	16.40	1656.267	0.0588
		$^3\mathrm{P}_2$	$^3\mathrm{P}^\mathrm{o}_2$	43.40	1657.008	0.104
N II	$2s^22p^2 - 2s2p^3$	$^3\mathrm{P}_0$	$^3\mathrm{D}^\mathrm{o}_1$	0	1083.990	0.115
		$^3\mathrm{P}_1$	$^3\mathrm{D}^\mathrm{o}_2$	48.7	1084.580	0.0861
		$^3\mathrm{P}_2$	$^3\mathrm{D}^\mathrm{o}_3$	130.8	1085.701	0.0957
N I	$2s^22p^3 - 2s^22p^23s$	$^4\mathrm{S}^\mathrm{o}_{3/2}$	$^4\mathrm{P}_{5/2}$	0	1199.550	0.130
		$^4\mathrm{S}^\mathrm{o}_{3/2}$	$^4\mathrm{P}_{3/2}$	0	1200.223	0.0862
O I	$2s^22p^4 - 2s^22p^33s$	$^3\mathrm{P}_2$	$^3\mathrm{S}^\mathrm{o}_1$	0	1302.168	0.0520
		$^3\mathrm{P}_1$	$^3\mathrm{S}^\mathrm{o}_1$	158.265	1304.858	0.0518
		$^3\mathrm{P}_0$	$^3\mathrm{S}^\mathrm{o}_1$	226.977	1306.029	0.0519
Mg II	$2p^63s - 2p^63p$	$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^\mathrm{o}_{1/2}$	0	2803.531	0.303
		$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^\mathrm{o}_{3/2}$	0	2796.352	0.608
Al III	$2p^63s - 2p^63p$	$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^\mathrm{o}_{1/2}$	0	1862.790	0.277
		$^2\mathrm{S}_{1/2}$	$^2\mathrm{P}^\mathrm{o}_{3/2}$	0	1854.716	0.557

Table 9.4 in [Draine]
See also Table 9.3

[3] Line Broadening Mechanisms

- ***Atomic levels are not infinitely sharp***, nor are the lines connecting them.
  - (1) Doppler (Thermal) Broadening
  - (2) Natural Broadening
  - (3) Collisional Broadening
  - (4) Thermal Doppler + Natural Broadening
- **[1] Doppler (Thermal) Broadening**
  - The simplest mechanism for line broadening in the Doppler effect. An atom is in thermal motion, so that the frequency of emission or absorption in its own frame corresponds to a different frequency for an observer.
  - Each atom has its own Doppler shift, so that the net effect is to spread the line out, but not to change its total strength.
  - The change in frequency associated with an atom with velocity component v_z along the photon propagation direction (say, z axis) is, to lowest order in v_z/c, given by

$$\nu - \nu_0 = \nu_0 \frac{v_z}{c}$$

Recall Doppler shift: $\left[\frac{\nu}{\nu_0} = \frac{1}{\gamma\left(1-\beta\cos\theta\right)} \quad \rightarrow \quad \nu \approx \nu_0\left(1+\beta\cos\theta\right) \quad \rightarrow \quad \nu - \nu_0 = \frac{\nu_0 v_z}{c}\right]$

  - Here, ν_0 is the rest-frame frequency.

- We need to consider the velocity distribution of atoms. The number of atoms having velocities in the range $(v_z,\ v_z + dv_z)$ is proportional to

$$f(v_z)dv_z = \left(\frac{m}{2\pi kT}\right)^{1/2} \exp\left(-\frac{mv_z^2}{2kT}\right) dv_z = \frac{1}{\sqrt{2\pi}v_{\rm rms}} \exp\left(-\frac{v_z^2}{2v_{\rm rms}^2}\right) dv_z$$

- From the Doppler shift formula, we have

Here, $m =$ mass of the atom

$$v_z = \frac{c(\nu - \nu_0)}{\nu_0} \quad \rightarrow \quad dv_z = \frac{cd\nu}{\nu_0}$$

- Therefore, the strength of the emission is proportional to

$$\exp\left(-\frac{mv_z^2}{2kT}\right) dv_z = \frac{c}{\nu_0} \exp\left[-\frac{mc^2(\nu-\nu_0)^2}{2\nu_0^2 kT}\right] d\nu$$

- Then, the normalized profile function is

$$\left(v_{\rm rms} = \sqrt{\frac{kT}{m}}\right)$$

$$\phi(\nu) = \frac{1}{\Delta\nu_{\rm D}\sqrt{\pi}} e^{-(\nu-\nu_0)^2/(\Delta\nu_{\rm D})^2}$$ where $\Delta\nu_{\rm D} = \nu_0 \frac{v_{\rm th}}{c}$ is the Doppler width.

$$= \frac{\nu_0}{c}\sqrt{\frac{2kT}{m}} = \nu_0 \frac{\sqrt{2}v_{\rm rms}}{c}$$

- Numerical value of the velocity broadening is

$$v_{\rm th} = \left(\frac{2k_{\rm B}T}{m}\right)^{1/2} = 1.3\,{\rm km\,s^{-1}}\left(\frac{T}{100\,{\rm K}}\right)^{1/2}\left(\frac{m}{m_{\rm H}}\right)^{-1/2}$$

- In addition to thermal motions, there can be turbulent velocities associated with macroscopic velocity fields. The turbulent motions are accounted for by an effective Doppler width.

$$\Delta\nu_{\rm D} = \nu_0 \frac{b}{c}$$

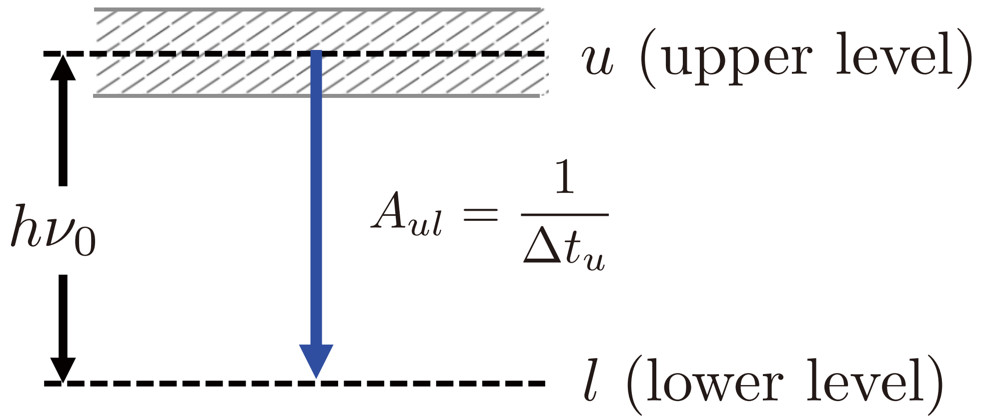
$$b \equiv \left(v_{\rm th}^2 + v_{\rm turb}^2\right)^{1/2}$$

Note that the convolution of two gaussian functions is a gaussian.

where $v_{\rm turb}$ is $\sqrt{2}$ times a root-mean-square measure of the turbulent velocities. This assumes that the turbulent velocities also have a Gaussian distribution.

- **[2] Natural Broadening**
  - The intrinsic line width of a line is due to ***the Heisenberg uncertainty principle.*** If an energy level u has a lifetime Δt, then uncertainty (spread) in energy ΔE must be $\Delta E \sim \hbar/\Delta t$ ($\hbar = h/2\pi$), and the resulting spread in the frequency of emitted photons is $\Delta\nu = \Delta E/h$.

(1) Line width due to the uncertainty principle:



A_{ul} = decay rate
= decay probability per unit time, Einstein A coefficient.

ΔE_u = uncertainty in energy of u

Δt_u = the uncertainty in time of occupation of u

$$\Delta\nu_u = \text{spread in frequency}$$
$$= \Delta E_u/h = 1/(2\pi\Delta t_u) = A_{ul}/(2\pi)$$

(2) Line width of the Lorentz function:

$$\phi_\nu = \frac{\gamma/4\pi^2}{(\nu-\nu_0)^2 + (\gamma/4\pi)^2}$$

In terms of the line width $\Delta\nu_u$, the normalized Lorentz profile can be rewritten as

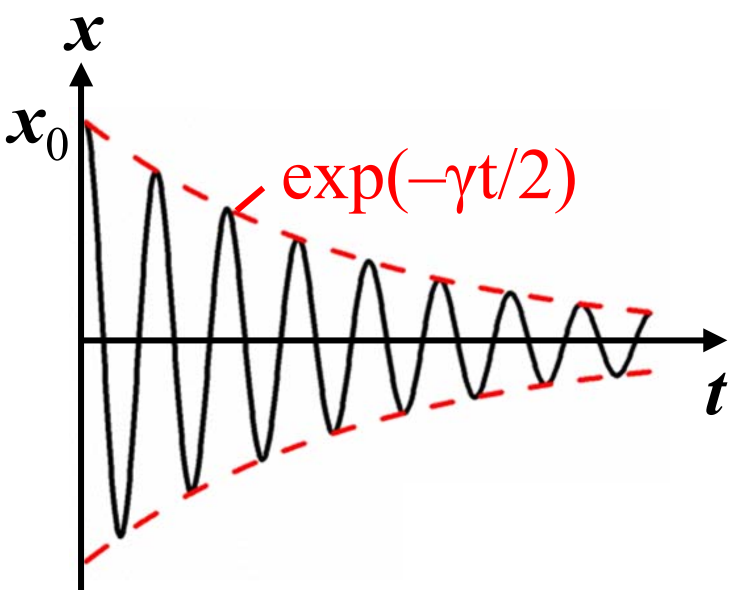
$$\phi_\nu = \frac{1}{2\pi}\frac{\Delta\nu_u/2}{(\nu-\nu_0)^2 + (\Delta\nu_u/2)^2}$$

Hence, the FWHM of the Lorentz function: $\Delta\nu_u = \gamma/2\pi$

Comparing (1) and (2), we find that γ is equivalent to the the Einstein A-coefficient., i.e., $\gamma = A_{ul}$.

Classical physical Meaning

Suppose that the electric field is of the form $e^{-\gamma t/2}$ and then the energy decays proportional to $e^{-\gamma t}$.



We then have an emitted spectrum determined by the decaying sinusoid type of electric field.

Its Fourier transform (spectral profile) is a Lorentz (or natural, or Cauchy) profile:

$$\phi_\nu = \frac{\gamma/4\pi^2}{(\nu - \nu_0)^2 + (\gamma/4\pi)^2}$$

Semiclassical (Weissokpf-Woolley) Picture of Quantum Levels

- In the semiclassical picture, each level is viewed as a continuous distribution of sublevels with energies close to the energy of the level (E_n).

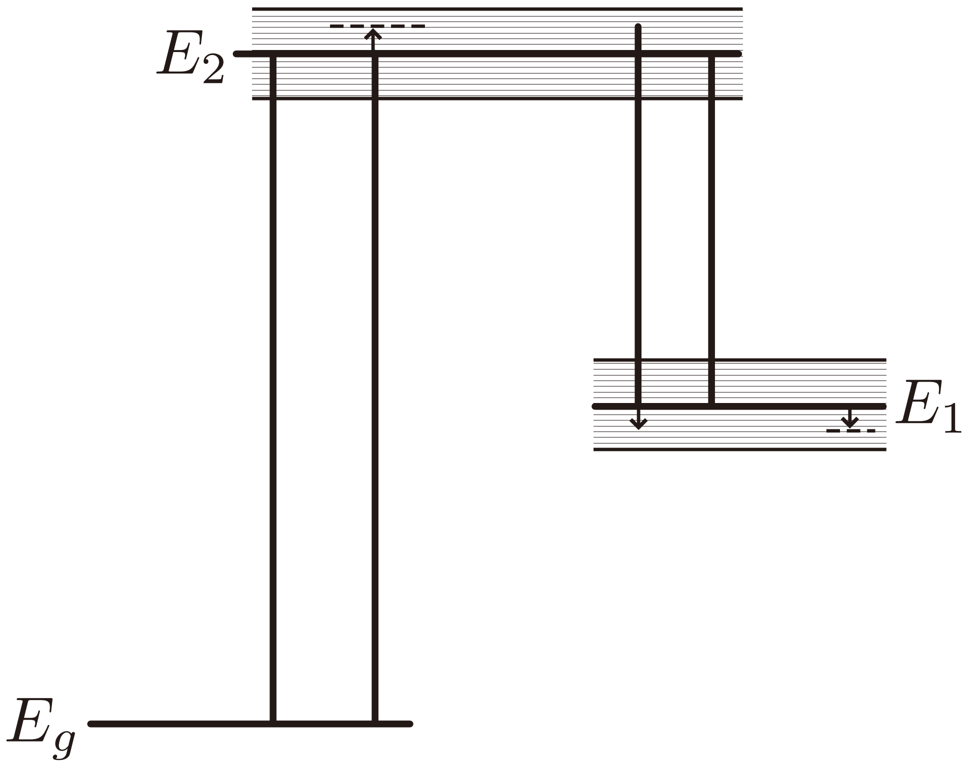
The distribution of sublevels are explained by the Heisenberg Uncertainty Principle. The level has a lifetime $\Delta t = 1/A$ (A = Einstein A coefficient) and a spread in energy about $\Delta E \approx \hbar/\Delta t = \hbar A$.

$$\Delta E \Delta t \approx \hbar$$

The ground level has no spread in energy

because $\Delta t = \infty$.

The atom is in a definite sublevel of some level.



A transition in a spectral line is considered to be an instantaneous transition between a definite sublevel of an initial level to a definite sublevel of a final level.

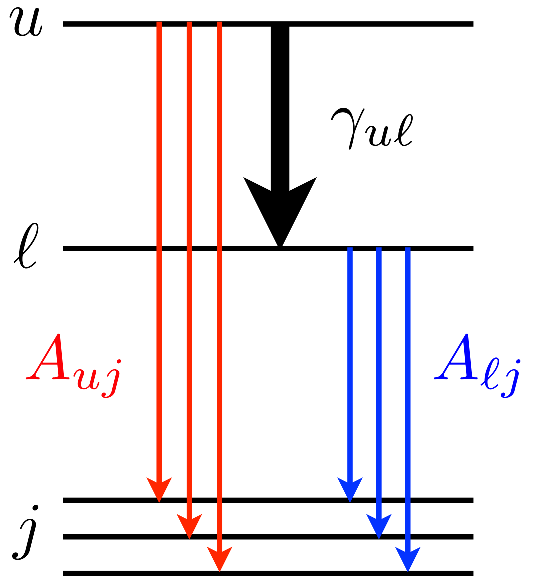
The energy spread of a sublevel is described by a Lorentzian profile.

- The intrinsic line width is $\gamma = A_{ul}$.
  - This means forbidden lines are intrinsically narrower than permitted lines.
  - For instance, the permitted Lyα line has $A_{u\ell}/\nu_{u\ell} \sim 3\times10^{-7}$, while the forbidden [OIII] 5007Å is has a tiny width $A_{u\ell}/\nu_{u\ell} \sim 3\times10^{-17}$.
  - The intrinsic line width of [O III] 5007Å is equivalent to the Doppler broadening of

$$\Delta\nu_{\rm D} = \nu_{u\ell}\frac{\Delta v}{c} \quad \longrightarrow \quad \Delta v \sim 3\times10^{-17}c \sim 10\,{\rm nm\,s^{-1}} \sim 30\,{\rm cm\,yr^{-1}}$$

- For a multiple-level absorber, the upper and lower can both be broadened by transitions to other levels.

$$\gamma_{u\ell} = \sum_{E_j<E_u} A_{uj} + \sum_{E_j<E_\ell} A_{\ell j}$$



  - For Lyα (n = 1-2), $\gamma_{u\ell} = A_{21} = 6.3\times10^{8}\,{\rm s^{-1}}$

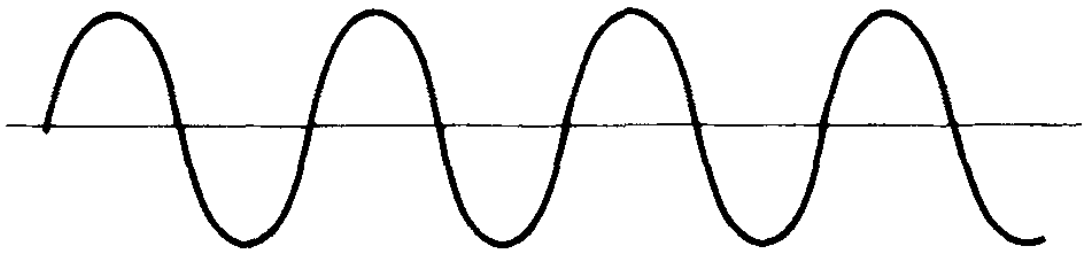
$$\Delta\nu/\nu \sim 4\times10^{-8}$$

  - For Hα (n = 2-3), $\gamma_{u\ell} = A_{32} + A_{31} + A_{21} = 8.9\times10^{8}\,{\rm s^{-1}}$

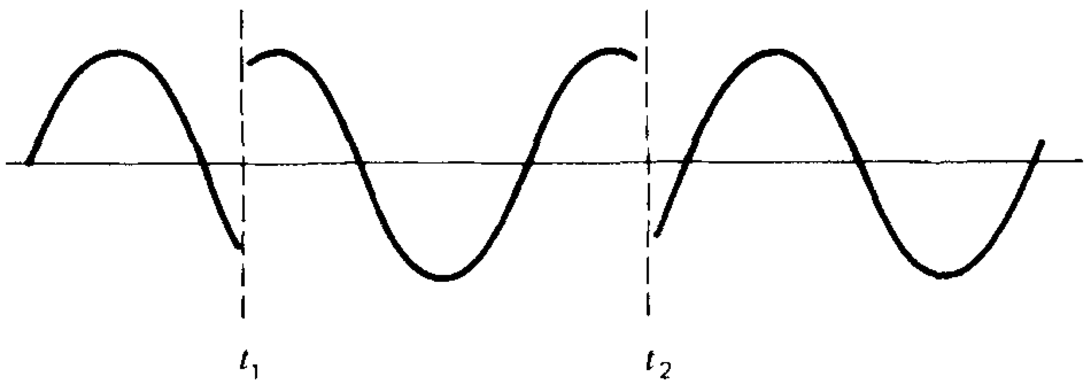
$$\Delta\nu/\nu \sim 3\times10^{-7}$$

- **[3] Collisional Broadening (or Pressure Broadening)**
  - The Lorentz profile applies even to certain types of collisional broadening mechanisms.
  - If the atom suffers *“elastic”* collisions with other particles while it is emitting, the phase of the emitted radiation can be altered suddenly. If the phase changes completely randomly at the collision times, then information about the emitting frequencies is lost.
  - If the collisions occur with frequency $\nu_{\rm col}$, that is, each atom experiences $\nu_{\rm col}$ collisions per unit time on the average, then the profile is

$$\phi_\nu = \frac{\Gamma/4\pi^2}{(\nu-\nu_0)^2+(\Gamma/4\pi)^2} \quad \text{where} \quad \Gamma = \gamma + 2\nu_{\rm col}$$



purely sinusoidal



random phase interruptions by atomic collisions

For derivation of the above formula, see Problem 10.7 of Radiative Processes in Astrophysics [Rybiki & Lightman], Chap. 11 of Atomic Spectroscopy and Radiative Processes [Degl’Innocenti], and Chap 8.3 in Theory of Stellar Atmospheres [Hubeny & Mihalas]

- **[4] Voigt profile : Thermal + Natural broadening**
  - Atoms shows both a Lorentz profile plus the Doppler effect. In this case, we can write the profile as an average of the Lorentz profile over the various velocity states of the atom. Let's assume that the photon propagates along the z-axis.

Change of variables for the Maxwell distribution: $v_{\rm th} \equiv \sqrt{\frac{2kT}{m}}, \; y \equiv \frac{v_z}{v_{\rm th}}$

$$f_{v_z} = \frac{1}{\pi^{1/2}\left(2kT/m\right)^{1/2}} \exp\left(-mv_z^2/2kT\right) \longrightarrow f_y = \frac{1}{\pi^{1/2}} \exp(-y^2)$$

To interact with an atom with velocity v_z , the photon central frequency should be $\nu_0 + \nu_0(v_z/c)$. Then, the Lorentz profile at the frequency $\nu' = \nu - [\nu_0 + \nu_0(v_z/c)] = (\nu - \nu_0) - \nu_0(v_{\rm th}/c)y$ is supposed to be multiplied with the Maxwell distribution.

Change of variables for the Lorentz function: $\phi_y^{\rm L} = \phi_\nu^{\rm L} \left|\frac{d\nu}{dy}\right| = \phi_\nu^{\rm L} \times \left(\nu_0 \frac{v_{\rm th}}{c}\right)$

Let $\Delta\nu_{\rm D} \equiv \nu_0 \frac{v_{\rm th}}{c}, \; u \equiv \frac{\nu - \nu_0}{\Delta\nu_{\rm D}} = \frac{\nu - \nu_0}{\nu_0} \frac{c}{v_{\rm th}}, \; a = \frac{\Gamma/4\pi}{\Delta\nu_{\rm D}}$

$$\begin{aligned}
\phi(\nu) &= \int_{-\infty}^{\infty} \phi_y^{\rm L} f_y dy \\
&= \int_{-\infty}^{\infty} \left(\nu_0 \frac{v_{\rm th}}{c}\right) \frac{\Gamma/4\pi^2}{\left[(\nu - \nu_0) - \nu_0(v_{\rm th}/c)y\right]^2 + (\Gamma/4\pi)^2} \left(\frac{1}{\pi^{1/2}}\right) \exp\left(-y^2\right) dy \\
&= \frac{a}{\pi^{3/2}\Delta\nu_{\rm D}} \int_{-\infty}^{\infty} \frac{e^{-y^2}}{(u - y)^2 + a^2} dy
\end{aligned}$$

- The profile can be written using the Voigt function.

$$\phi(\nu) = \frac{1}{\Delta\nu_{\rm D}\sqrt{\pi}} H(u,a)$$

Here, a is a ratio of the intrinsic broadening to the thermal broadening.

u is a measure of how far you are from the line center, in units of thermal broadening parameter.

In terms of Doppler velocity, u can be expressed as

$$u = \frac{\nu - \nu_0}{\Delta\nu_{\rm D}} = \frac{\nu - \nu_0}{\nu_0}\frac{c}{v_{\rm th}}$$

In the velocity term,

$$u = \frac{v}{v_{\rm th}}, \quad \text{where } v = \frac{\nu - \nu_0}{\nu_0} c$$

Voigt-Hjerting function:

$$H(u,a) \equiv \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{e^{-y^2}dy}{(u-y)^2 + a^2}$$

$$a \equiv \frac{\Gamma}{4\pi\Delta\nu_{\rm D}}$$

$$u \equiv \frac{\nu - \nu_0}{\Delta\nu_{\rm D}}$$

$$\Delta\nu_{\rm D} = \frac{\nu_0}{c}\sqrt{\frac{2kT}{m}}$$

Including the turbulent motion,

$$\Delta\nu_{\rm D} = \nu_0 \frac{v_{\rm th}}{c} \quad \rightarrow \quad \Delta\nu_{\rm D} = \nu_0 \frac{b}{c} = \frac{b}{\lambda_0}$$

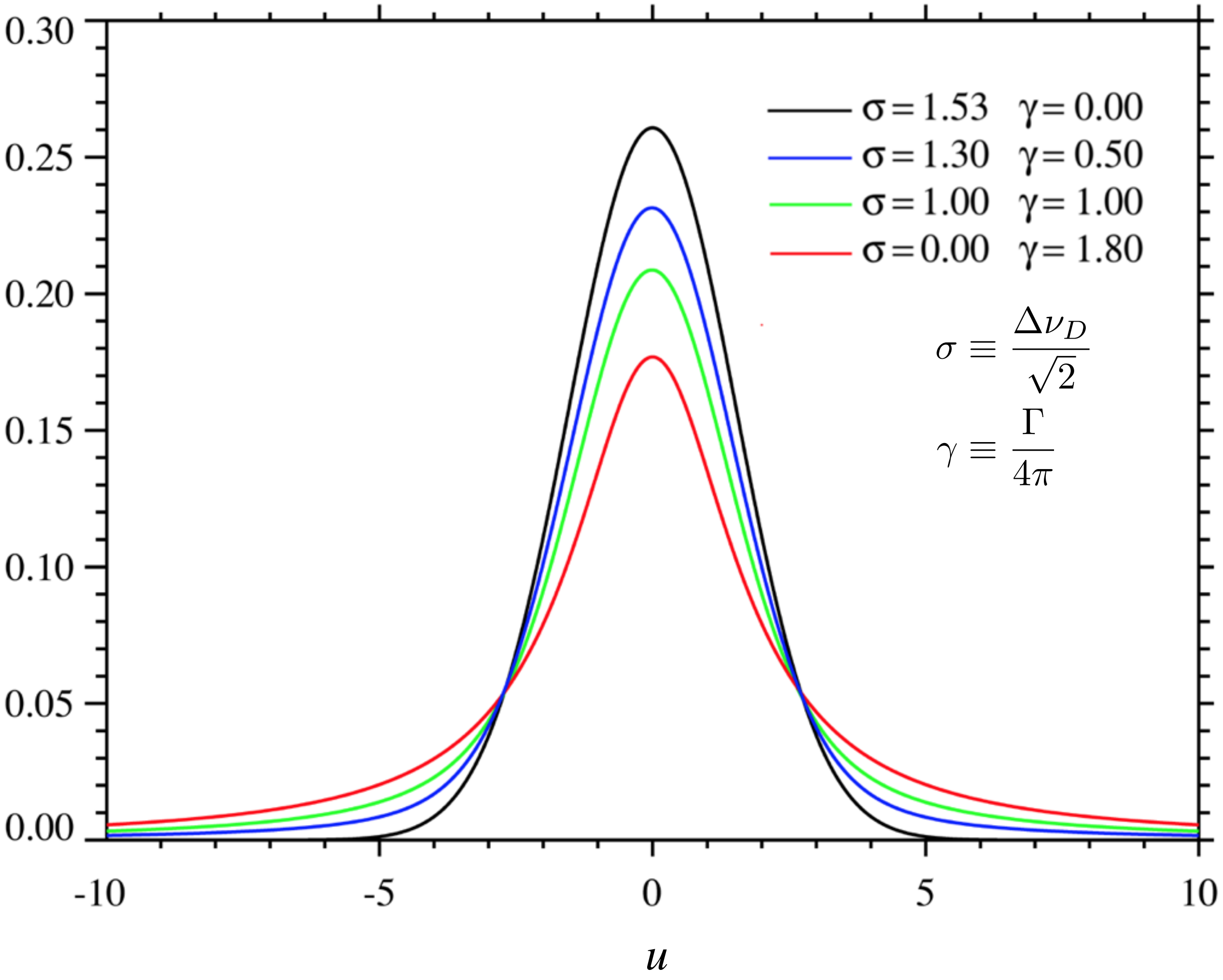
$$\text{where } b = \sqrt{v_{\rm th}^2 + v_{\rm turb}^2}, \quad v_{\rm th} = \sqrt{\frac{2kT}{m}}$$

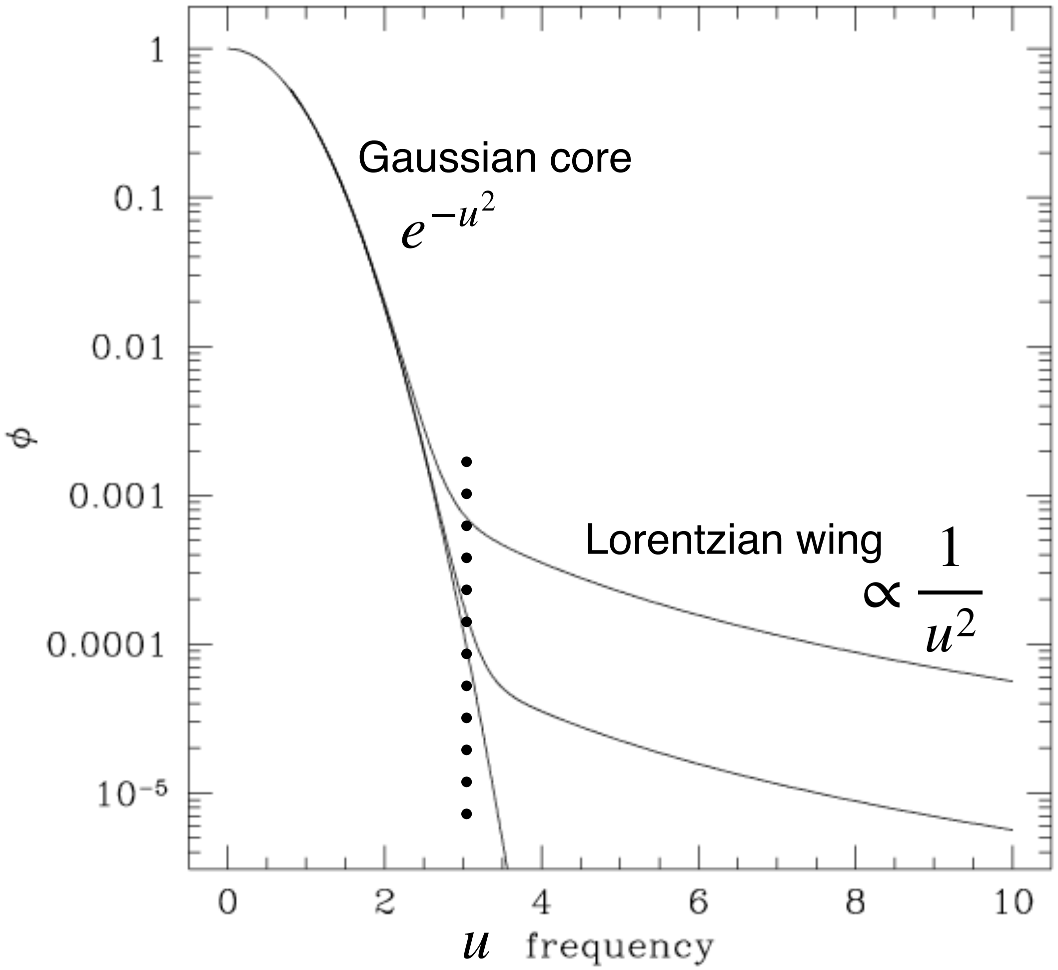
$$u = \frac{v}{b}$$

Properties of Voigt Function

- For small a, the “core” of the line is dominated by the Gaussian (Doppler) profile, whereas the “wings” are dominated by the Lorentz profile.

$$H(u,a) \equiv \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{e^{-y^2}dy}{(u-y)^2+a^2}$$





- In most case, $a \ll 1$. For Lyα at T = 100 K, $a \sim 0.05$.

- Line center:

$$H(0,a) = \exp\left(a^2\right) \mathrm{Erfc}\left(a\right) \approx 1 - \frac{2}{\sqrt{\pi}} a + a^2 - \mathcal{O}(a^3)$$

- Taylor series expansion of the Voigt function :

$$H(u,a) \equiv \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2} dy}{(u-y)^2 + a^2}$$

  - Near the line center ($u \rightarrow 0$), the contribution to the integral is dominated by $y = u$. Therefore,

$$H(u,a) \simeq \frac{a}{\pi} e^{-u^2} \int_{-\infty}^{\infty} \frac{dy}{y^2 + a^2} = e^{-u^2}$$

which is known as the Doppler core.

  - In the line wings away from the core ($u \gg 1$), the integral is dominated by $y \sim 0$ because of the rapidly decreasing function in the numerator.

$$H(u,a) \simeq \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2} dy}{u^2} = \frac{a}{\pi} \frac{\sqrt{\pi}}{u^2} = \frac{a}{\sqrt{\pi} u^2}$$

- In summary, we obtain the Voigt function in a Taylor series expansion around $a = 0$.

$$H(u,a) \approx H(u,0) + a\,\frac{dH}{da}\bigg|_{a=0} \approx e^{-u^2} + a\frac{1}{\sqrt{\pi}u^2}$$

  - The first term represents the Gaussian core, provided by the thermal broadening, and the second term represents the Lorentizan damping wing.

- Transition from Doppler core to damping wing can be found by solving:

$$e^{u^2} = \frac{\sqrt{\pi}}{a}u^2 \quad \rightarrow \quad u^2 = \ln\left(\frac{\sqrt{\pi}}{a}\right) + \ln u^2$$

  - The solution for this transcendental equation for Lyα is

for hydrogen

$$b = 13\,\mathrm{km\,s^{-1}}\left(T/10^4\,\mathrm{K}\right)^{1/2}$$

$$u^2 \approx 10.31 + \ln\left[\left(\frac{6.265\times10^8\,\mathrm{s}^{-1}}{\gamma_{u\ell}}\right)\left(\frac{1215.67\,\text{Å}}{\lambda_{u\ell}}\right)\left(\frac{b}{10\,\mathrm{km\,s^{-1}}}\right)\right]$$

provided that the quantity in square brackets is not very large or very small. The damping wing for $|u| \gtrsim 3.2$ or velocity shifts $|v| \gtrsim 3.2\ (b/10\,\mathrm{km\,s^{-1}})$.

Voigt Line Profile

- The profile can be written using the Voigt function.

$$\phi(\nu) = \frac{1}{\Delta\nu_{\rm D}\sqrt{\pi}} H(u,a)$$

Here, a is a ratio of the intrinsic broadening to the thermal broadening.

u is a measure of how far you are from the line center, in units of thermal broadening parameter.

In terms of Doppler velocity, u can be expressed as

$$u = \frac{\nu - \nu_0}{\Delta\nu_{\rm D}} = \frac{\nu - \nu_0}{\nu_0}\frac{c}{v_{\rm th}}$$

In the velocity term,

$$u = \frac{v}{v_{\rm th}}, \quad \text{where } v = \frac{\nu - \nu_0}{\nu_0} c$$

Voigt-Hjerting function:

$$H(u,a) \equiv \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{e^{-y^2}dy}{(u-y)^2 + a^2}$$

$$a \equiv \frac{\Gamma}{4\pi\Delta\nu_{\rm D}}$$

$$u \equiv \frac{\nu - \nu_0}{\Delta\nu_{\rm D}}$$

$$\Delta\nu_{\rm D} = \frac{\nu_0}{c}\sqrt{\frac{2kT}{m}}$$

Including the turbulent motion,

$$\Delta\nu_{\rm D} = \nu_0 \frac{v_{\rm th}}{c} \quad \rightarrow \quad \Delta\nu_{\rm D} = \nu_0 \frac{b}{c} = \frac{b}{\lambda_0}$$

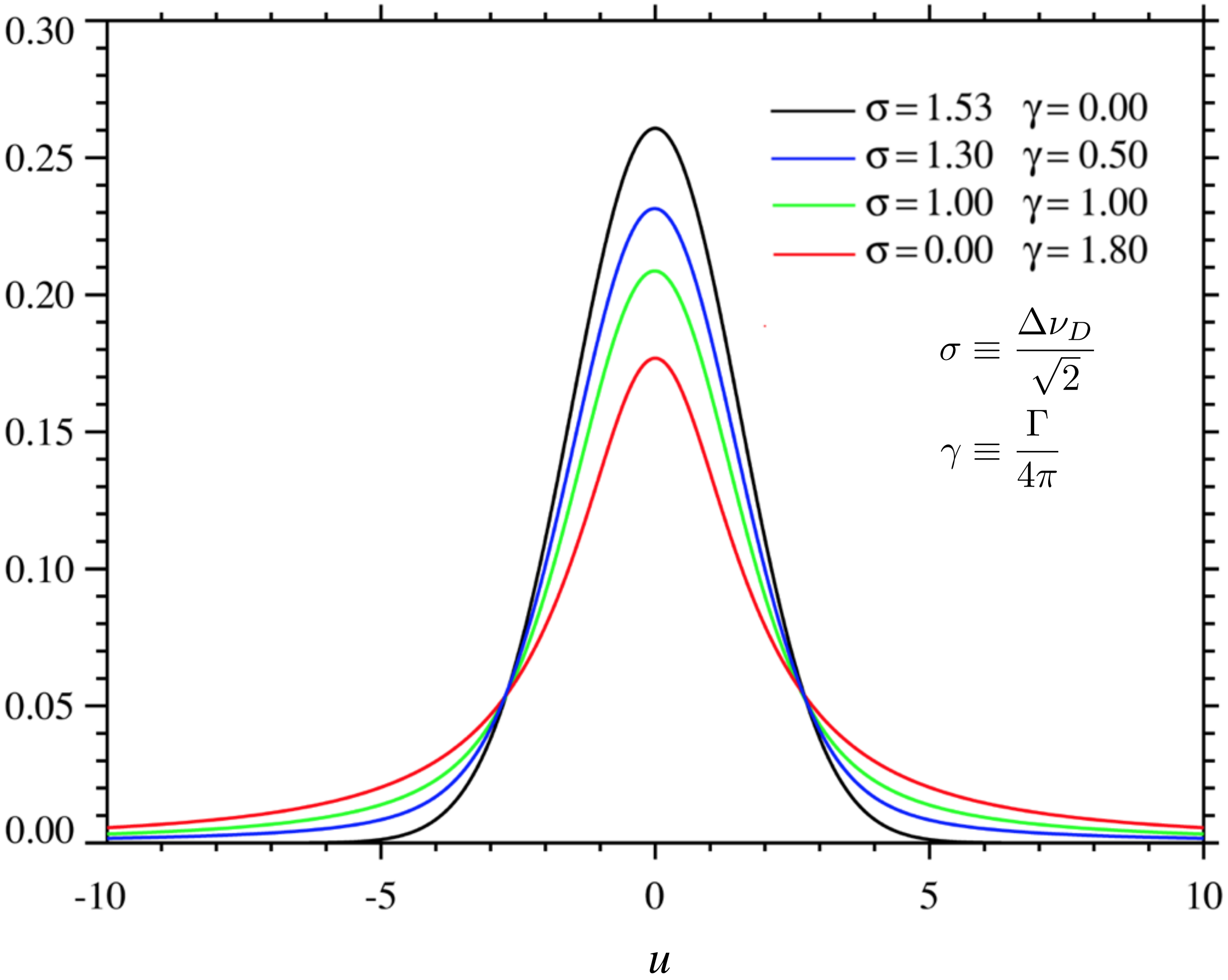
$$\text{where } b = \sqrt{v_{\rm th}^2 + v_{\rm turb}^2}, \quad v_{\rm th} = \sqrt{\frac{2kT}{m}}$$

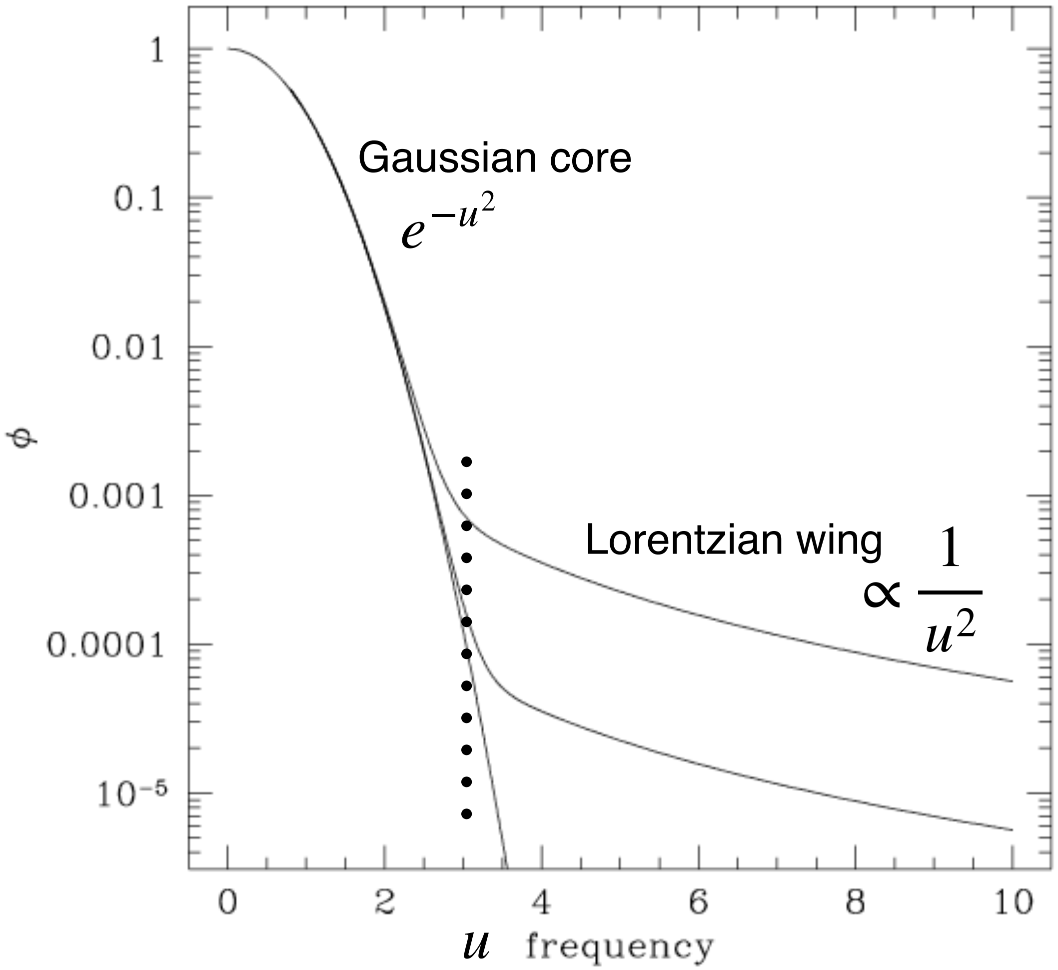
$$u = \frac{v}{b}$$

Properties of Voigt Function

- For small a, the “core” of the line is dominated by the Gaussian (Doppler) profile, whereas the “wings” are dominated by the Lorentz profile.

$$H(u,a) \equiv \frac{a}{\pi}\int_{-\infty}^{\infty}\frac{e^{-y^2}dy}{(u-y)^2+a^2}$$





- In most case, $a \ll 1$. For Lyα at T = 100 K, $a \sim 0.05$.

- Line center:

$$H(0,a) = \exp\left(a^2\right) \mathrm{Erfc}\,(a) \approx 1 - \frac{2}{\sqrt{\pi}}a + a^2 - \mathcal{O}(a^3)$$

- Taylor series expansion of the Voigt function :

$$H(u,a) \equiv \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{e^{-y^2}dy}{(u-y)^2 + a^2}$$

  - Near the line center ($u \to 0$), the contribution to the integral is dominated by $y = u$. Therefore,

$$H(u,a) \simeq \frac{a}{\pi}e^{-u^2}\int_{-\infty}^{\infty} \frac{dy}{y^2 + a^2} = e^{-u^2}$$

    which is known as the Doppler core.

  - In the line wings away from the core ($u \gg 1$), the integral is dominated by $y \sim 0$ because of the rapidly decreasing function in the numerator.

$$H(u,a) \simeq \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{e^{-y^2}dy}{u^2} = \frac{a}{\pi}\frac{\sqrt{\pi}}{u^2} = \frac{a}{\sqrt{\pi}u^2}$$

- In summary, we obtain the Voigt function in a Taylor series expansion around $a = 0$.

$$\boxed{H(u,a) \approx H(u,0) + a\,\frac{dH}{da}\bigg|_{a=0} \approx e^{-u^2} + a\frac{1}{\sqrt{\pi}u^2}}$$

  - The first term represents the Gaussian core, provided by the thermal broadening, and the second term represents the Lorentizan damping wing.

- Transition from Doppler core to damping wing can be found by solving:

$$e^{u^2} = \frac{\sqrt{\pi}}{a}u^2 \quad \rightarrow \quad u^2 = \ln\left(\frac{\sqrt{\pi}}{a}\right) + \ln u^2$$

for hydrogen

$$b = 13\,\mathrm{km\,s^{-1}}\,\left(T/10^4\,\mathrm{K}\right)^{1/2}$$

  - The solution for this transcendental equation for Lyα is

$$\boxed{u^2 \approx 10.31 + \ln\left[\left(\frac{6.265\times10^8\,\mathrm{s^{-1}}}{\gamma_{u\ell}}\right)\left(\frac{1215.67\,\text{Å}}{\lambda_{u\ell}}\right)\left(\frac{b}{10\,\mathrm{km\,s^{-1}}}\right)\right]}$$

provided that the quantity in square brackets is not very large or very small. The damping wing for $|u| \gtrsim 3.2$ or velocity shifts $|v| \gtrsim 3.2\ (b/10\,\mathrm{km\,s^{-1}})$.

[Q5] Voigt profile

- We want to derive an approximate formula for the transition point from the Gaussian core to the Lorentz wing, which is defined by

$$u^2 = \ln\left(\sqrt{\pi}/a\right) + \ln u^2 \qquad \text{or} \qquad x = \ln\left(\sqrt{\pi}/a\right) + \ln x, \ \text{ where } x \equiv u^2$$

The above equation can be expressed in the form:

$$x = g(x) \ \text{ where } g(x) = \ln x + \ln\left(\sqrt{\pi}/a\right)$$

This equation can be solved using ***"Fixed Point Iteration Method."*** Starting from any initial point x_0 , the following recursive process gives an approximate solution of the equation.

$$x_{n+1} = g(x_n)$$

(1) Find a numerical solution x_* for Lyα line with $b = 10\,\mathrm{km\,s^{-1}}$, which is appropriate for Lyα in the WNM with $T \sim 10000$ K.

(2) Let's denote the width parameter as a_* for $b = 10\,\mathrm{km\,s^{-1}}$. This means that

$$x_* = \ln x_* + \ln\left(\sqrt{\pi}/a_*\right)$$

Now, for any parameter a which is different from a_*, you may express the constant term in as follows: $g(x)$

$$\ln\left(\sqrt{\pi}/a\right) = \ln\left(a_*/a\right) + \ln\left(\sqrt{\pi}/a_*\right)$$

To find the solution for $a \neq a_*$ $(\mathrm{but,}\ \ a \approx a_*)$, choose an initial guess to be $x_0 = x_*$. Show that the solution for any a can be expressed as (after only a single iteration):

$$x_1 = x_* + \ln\left(a_*/a\right)$$

Insert numerical values into the above equation and compare it with Eq. (2.39) in Ryden's book (our textbook).

(3) Insert numerical values into the above equation compare it with the results in this lecture note and Eq. (6.42) in Draine's book.