Interstellar Medium (ISM)

Week 10
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선광일 (Kwangil Seon)
KASI / UST

Intermediate States - expansion phase

- Assumptions:
  - The shocked gas layer is thin.
  - The ionization front follows the shock front and the expansion velocity of ionized sphere is approximately the same as the shock velocity.

$$V_{\mathrm{I-front}} \approx V_{\mathrm{shock}} \qquad \frac{dR}{dt} = V_{\mathrm{s}}$$

- Expansion:
  - The pressure behind a strong “isothermal” shock (high Mach number) is related to the shock velocity:

$$P_{\mathrm{s}} = \rho_0 V_{\mathrm{s}}^2 = n_0 m_{\mathrm{H}} V_s^2$$

  - Now assume that the pressure behind the shock wave is equal to the pressure of the ionized gas (pressure equilibrium).

$$P_{\mathrm{i}} = 2n_{\mathrm{i}}kT = n_{\mathrm{i}} m_{\mathrm{H}} c_{\mathrm{i}}^2 \quad \left(c_{\mathrm{i}}^2 \equiv \frac{2kT}{m_{\mathrm{H}}} \right)$$

for fully-ionized hydrogen gas

  - Then, the shock velocity is given by

$$P_{\mathrm{s}} = P_{\mathrm{i}} \quad \rightarrow \quad V_{\mathrm{s}}^2 = \frac{n_{\mathrm{i}}}{n_0} c_{\mathrm{i}}^2 \quad \rightarrow \quad \frac{V_{\mathrm{s}}^2}{c_{\mathrm{i}}^2} = \frac{n_{\mathrm{i}}}{n_0}$$

- We assume that the amount of fresh neutral gas to be ionized is very small. Then, the ionization balance for the region within R gives

$$Q_0 = \frac{4\pi}{3} R^3 n_{\rm i}^2 \alpha_{\rm B} \quad \rightarrow \quad R^3 = \frac{3Q_0}{4\pi n_{\rm i}^2 \alpha_{\rm B}} = R_s^3 \left(\frac{n_0}{n_{\rm i}}\right)^2$$

R_s = Strömgren radius for the initial stage.

- Combining with $\frac{V_{\rm s}^2}{c_{\rm i}^2} = \frac{n_{\rm i}}{n_0}$, the equation for the expansion of the ionization front is

$$R^3 = R_s^3 \left(\frac{c_{\rm i}}{V_{\rm s}}\right)^4 \qquad \longleftarrow \quad \frac{dR}{dt} = V_{\rm s}$$

$$\rho \equiv R/R_s, \quad \tau \equiv c_{\rm i} t/R_s \quad \longrightarrow \quad \rho^3 \left(\frac{d\rho}{d\tau}\right)^4 = 1 \quad \rightarrow \quad \boxed{\rho^{3/4} \frac{d\rho}{d\tau} = 1}$$

- For a suitable boundary condition, we assume that the initial Strömgren sphere is set up at τ_0 (a very small fraction of the lifetime of the H II region):

$$R = R_s \text{ at } \tau = \tau_0$$

Then, the solution of the differential equation is

$$\rho = \left[1 + \frac{7}{4}(\tau - \tau_0)\right]^{4/7} \qquad R = R_s \left(1 + \frac{7}{4} \frac{t - t_0}{R_s/c_{\rm i}}\right)^{4/7}$$

- Expanding velocity is

$$\frac{dR}{dt} = c_{\mathrm{i}} \left(1 + \frac{7}{4} \frac{t - t_0}{R_s/c_{\mathrm{i}}}\right)^{-3/7}$$

- What is the time scale to reach the pressure equilibrium?

$$R(t_{\mathrm{eq}}) = R_{\mathrm{f}} \approx 34 R_s$$

$$R_s \left(1 + \frac{7}{4} \frac{t_{\mathrm{eq}}}{R_s/c_{\mathrm{i}}}\right)^{4/7} \approx 34 R_s$$

$$t_{\mathrm{eq}} \approx 273\,(R_s/c_{\mathrm{i}})$$

- The expanding velocity at this point is:

$$\boxed{V_{\mathrm{s}} = \frac{dR}{dt} = 0.71\, c_{\mathrm{i}} \quad \text{at} \quad t_{\mathrm{eq}} = 273 R_s/c_{\mathrm{i}}}$$

Timescales for typical HII region

- Let’s examine the case of an O7V star with

$$Q_0 = 10^{49}\,\mathrm{s}^{-1},\quad n_0 = 10^2\,\mathrm{cm}^{-3},\quad T = 10^4\,\mathrm{K}$$

  - Initial state: recombination time scale $t_\mathrm{rec} = (n_0\alpha_\mathrm{B})^{-1}$

$$R \approx R_s \text{ at } t = t_\mathrm{rec}$$
$$R_s \approx 3\,\mathrm{pc}\ (\approx 10^{19}\,\mathrm{cm})$$
$$t_\mathrm{rec} \approx 1000\,\mathrm{yr}$$

$$t \lesssim t_\mathrm{rec}$$
$$R(t) = R_s\left(1 - e^{-t/t_\mathrm{rec}}\right)^{1/3}$$
$$\frac{dR}{dt} = \frac{R_s}{3t_\mathrm{rec}}\frac{e^{-t/t_\mathrm{rec}}}{(1-e^{-t/t_\mathrm{rec}})^{2/3}}$$

  - Expansion phase: expansion timescale $t_\mathrm{exp} = R_s/c_\mathrm{i}$

$$\text{expansion velocity}: V_\mathrm{s} \le 0.65\,c_\mathrm{i} \text{ at } t \ge t_\mathrm{exp}$$
$$c_\mathrm{i} \approx 10\,\mathrm{km\,s^{-1}}$$
$$t_\mathrm{exp} \approx 3\times10^5\,\mathrm{yr} \quad\rightarrow\quad t_\mathrm{exp} \approx 200\,t_\mathrm{rec}$$

$$t_\mathrm{rec} \lesssim t \lesssim t_\mathrm{eq}$$
$$R = R_s\left(1 + \frac{7}{4}\frac{t-t_0}{R_s/c_\mathrm{i}}\right)^{4/7}$$
$$\frac{dR}{dt} = c_\mathrm{i}\left(1 + \frac{7}{4}\frac{t-t_0}{R_s/c_\mathrm{i}}\right)^{-3/7}$$

  - Final state: equilibrium timescale $t_\mathrm{eq} \approx 273R_s/c_\mathrm{i}$ (from expansion phase model)

$$R = R_\mathrm{f} \text{ at } t = t_\mathrm{eq}$$
$$R_\mathrm{f}/R_s \approx 34$$
$$t_\mathrm{eq} \approx 10^8\,\mathrm{yr} \quad\rightarrow\quad t_\mathrm{eq} \approx 300\,t_\mathrm{exp}$$

Does the Stromgren sphere reach pressure equilibrium?

- Main-sequence lifetime of an ionizing star

$$t_{\rm MS} \approx 10^{10} \left(\frac{M}{M_\odot}\right)^{-2} \,{\rm yr} \qquad\qquad t_{\rm MS} \approx 10^7 \,{\rm yr} \;\; {\rm for} \;\; M \approx 15 M_\odot$$

- Size
  - During the lifetime of an O star, which is less than 10 Myr, interstellar gas moving at 10 km/s will travel less than 100 pc, which is comparable with the diameter of the larger H II regions.
  - Thus, *before an H II region has expanded very far, its central energy source will be extinguished*.
- Time Scale:
  - Main-sequence lifetime of an ionizing star is 10 times smaller than the time scale for the pressure equilibrium:

$$\boxed{t_{\rm MS} \approx 10^7 \,{\rm yr} \ll t_{\rm eq} \approx 10^8 \,{\rm yr}}$$

  - ***It is unlikely that the final state (pressure equilibrium) of H II region can be reached during lifetime of star.***

Gas Dynamics

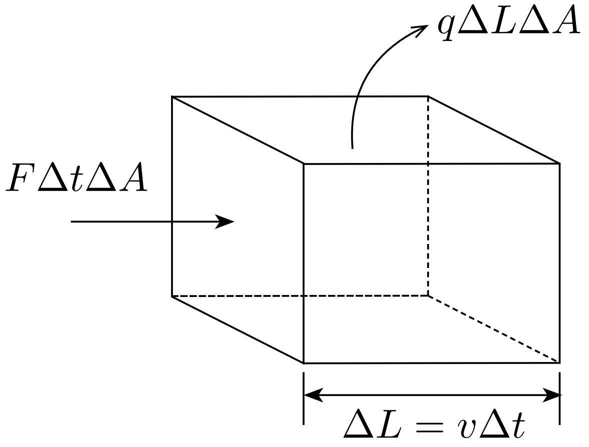
- Gas Dynamics / Shock

Introduction to Gas Dynamics

- Assumption for hydrodynamics:
  - particle mean free path << size of the region
  - We will describe the equations for conservation of mass, momentum and energy, in 1D space.
- ***Definition***
  - **Flux of a hydrodynamic quantity** q (for instance, density):

    Fluid moves a distance ΔL during a time interval Δt with a velocity v.

$$F \Delta t \Delta A = q \Delta L \Delta A \quad \rightarrow \quad F = qv$$

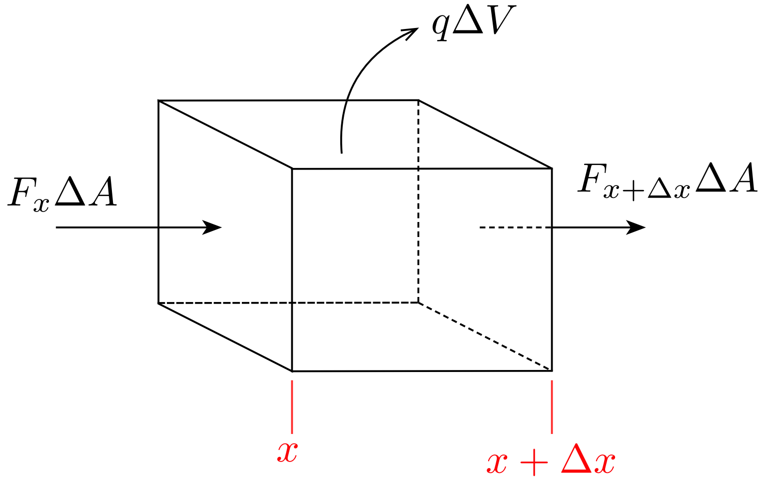


- ***Conservation equation for a quantity*** q
  - change of the quantity within a volume ΔV for a time interval Δt:

    Here, Δt and Δx are independent.

$$\frac{q\Delta V|_{t+\Delta t} - q\Delta V|_t}{\Delta t} = F\Delta A|_x - F\Delta A|_{x+\Delta x}$$

$$\frac{\partial q}{\partial t} = -\frac{\partial F}{\partial x} \longrightarrow \frac{\partial q}{\partial t} = -\frac{\partial (qv)}{\partial x}$$



  - Here, no sources or sinks of the quantity within ΔV were assumed. If any, the loss and gain terms should be added in the right-had side.

Mass Conservation

- Conservation equations
  - ***Mass conservation (continuity equation)***
    - mass within a volume dV = ρdV
    - no sources or sinks of material within dV
    - Consider the mass per unit area (dA), contained in the volume

$$\rho dV/dA = \rho dx \longrightarrow \frac{\partial}{\partial t}(\rho dx) = \overbrace{\rho u}^{\text{incoming}} - \overbrace{(\rho + d\rho)(u + du)}^{\text{outgoing}}$$

$$= -(\rho du + u d\rho + \cancel{d\rho du})$$

$$\boxed{\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x}}$$

- Mass loss and gain terms should be added in the right-had side, if necessary.

Momentum Conservation

- *Momentum conservation (Euler's equation)*

- momentum within dV (per unit area) = $(\rho dV)u/dA = \rho dxu$

  = change of momentum due to fluid flow and gas pressure acting on the surface of dV

$$\frac{\partial}{\partial t}(\rho u dx) = \overbrace{\rho u^2}^{\text{incoming}} - \overbrace{(\rho + d\rho)(u+du)^2}^{\text{outgoing}} + \overbrace{P}^{\text{incoming}} - \overbrace{P + dP}^{\text{outgoing}}$$

$$= \rho u^2 - \left(\rho u^2 + 2\rho u du + \cancel{\rho du^2} + u^2 d\rho + \cancel{2ud\rho du + d\rho du^2}\right) - dP$$

$$\frac{\partial}{\partial t}(\rho u) = -2\rho u \frac{\partial u}{\partial x} - u^2 \frac{\partial \rho}{\partial x} - \frac{\partial P}{\partial x}$$

$$\rho \frac{\partial u}{\partial t} + u \frac{\partial \rho}{\partial t} = -\rho u \frac{\partial u}{\partial x} - u\left(\rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x}\right) - \frac{\partial P}{\partial x}$$

Using mass conservation, $\frac{\partial u}{\partial t} = -\frac{\partial (\rho u)}{\partial x}$

$$\rho \frac{\partial u}{\partial t} = -\rho u \frac{\partial u}{\partial x} - \frac{\partial P}{\partial x}$$

or

$$\frac{\partial}{\partial t}(\rho u) = -2\rho u \frac{\partial u}{\partial x} - u^2 \frac{\partial \rho}{\partial x} - \frac{\partial P}{\partial x}$$

$$= -\frac{\partial}{\partial x}(\rho u^2) - \frac{\partial P}{\partial x}$$

$$\frac{\partial}{\partial t}(\rho u) = -\frac{\partial}{\partial x}\left(\rho u^2 + P\right)$$

- Further terms could be added in the right-hand side, accounting for forces due to gravity, magnetic fields, radiation field, and viscosity.

- The following quantity is sometimes known as **Bernoulli's constant**.

$$\boxed{\rho u^2 + P}$$

One may use it to understand why, for example, fast winds engulfing a house causes it to **explode**, rather than implode, because the pressure external to the house becomes lower than its value inside it.

- Viscous force is due to "internal friction" in the fluid (resistivity of the fluid to the flow), as two adjacent fluid parcels move relative to each other.)

$$\text{viscous force} \propto \frac{\partial^2 u}{\partial x^2}$$

The viscous force is usually much smaller than force due to gas pressure, but important in high-speed flows with large velocity gradients, as in accretion disks.

Gas Dynamics - Energy Conservation

- *Energy conservation*

- The first law of thermodynamics states that

  heat added in a system = change in internal energy + work done on surroundings

$$dQ = dU + PdV$$

- Internal energy (per particle) for ideal gas is

$$\begin{aligned} U/N &= \frac{3}{2}kT \quad \text{for monatomic gas (translation about 3 axes)} \\ U/N &= \frac{5}{2}kT \quad \text{for diatomic gas} \quad \text{(+rotation about 2 axes)} \\ U/N &= 3kT \quad \text{for polyatomic gas (+rotation about 3 axes)} \end{aligned}$$

Here, N is the number of particles.

An ideal gas is a theoretical gas composed of many randomly moving point particles whose only interactions are perfectly elastic collisions (no viscosity or heat conduction).

- In general, the internal energy per particle is

$$U/N = \frac{f}{2}kT \quad (f = \text{degree of freedom})$$

At high temperature, molecules have access to an increasing number of vibrational degrees of freedom, as they start to bend and stretch.

- The ideal gas law (the equation of state) for a perfect Maxwellian distribution.

$$\boxed{PV = NkT} \qquad P = \frac{N}{V}kT$$

- ***Specific heat capacity*** is the amount of ***heat energy required to raise the temperature of a material per unit of mass***.

  - specific heat capacity **at constant volume**:

$$c_V \equiv \frac{1}{M}\left(\frac{\partial Q}{\partial T}\right)_V = \frac{1}{M}\left(\frac{\partial U}{\partial T}\right)_V \qquad \boxed{c_V = \frac{f}{2}\frac{k}{m}}$$

$M = \text{total mass}$

$m = M/N = \text{mass per particle}$

$m = \mu m_{\rm H}$

$(\mu = \text{mean atomic weight per particle})$

  - specific heat capacity **at constant pressure**:

$$c_P \equiv \frac{1}{M}\left(\frac{\partial Q}{\partial T}\right)_P = \frac{1}{M}\left(\frac{\partial U}{\partial T}\right)_P + \frac{P}{M}\left(\frac{\partial V}{\partial T}\right)_P = \frac{1}{M}\frac{f}{2}Nk + \frac{P}{M}\frac{Nk}{P}$$

$$\boxed{\therefore \quad c_P = \frac{f+2}{2}\frac{k}{m} = c_V + \frac{k}{m}}$$

  - Ratio of specific heat capacities:

$$\gamma \equiv \frac{c_P}{c_V} = \frac{f+2}{f} = \frac{5}{3} \quad \text{for monatomic gas}$$
$$= \frac{7}{5} \quad \text{for diatomic gas}$$
$$= \frac{4}{3} \quad \text{for polyatomic gas}$$

γ is called the *adiabatic index*.

$c_P > c_V$

This inequality implies that when pressure is held constant, some of the added heat goes into PdV work instead of into internal energy.

- Energy Conservation - limiting cases
  - ***Adiabatic flow*** - negligible heat transport (Internal energy is changed only by work).

$$dQ = dU + PdV = Mc_V dT + PdV$$
$$dQ = 0$$
$$\rightarrow PdV = -Mc_V dT$$

$$PV = NkT$$
$$\rightarrow VdP + PdV = NkdT$$

We combine two equation and eliminate dT term:

$$\begin{aligned} VdP + PdV &= -\frac{Nk}{Mc_V}PdV \\ &= -\frac{k}{m}\frac{1}{c_V}PdV \end{aligned} \quad \longrightarrow \quad \begin{aligned} VdP &= -\left(1 + \frac{k}{m}\frac{1}{c_V}\right)PdV \\ &= -\frac{1}{c_V}\left(c_V + \frac{k}{m}\right)PdV \\ &= -\gamma PdV \end{aligned}$$

$$\longrightarrow \quad \frac{dP}{P} = -\gamma\frac{dV}{V}$$

We can rewrite this in terms of density:

$$\rho V = M$$
$$\rightarrow \rho dV + Vd\rho = 0 \quad \longrightarrow \quad \frac{dP}{P} = \gamma\frac{d\rho}{\rho}$$
$$\rightarrow \frac{d\rho}{\rho} = -\frac{dV}{V}$$

In summary,

$$\frac{dP}{P} = -\gamma\frac{dV}{V} \qquad P \propto \rho^{\gamma}$$
$$\frac{dP}{P} = \gamma\frac{d\rho}{\rho} \qquad P \propto V^{-\gamma}$$
$$\rightarrow T \propto V^{-(\gamma-1)}$$

adiabatic heating/cooling

- ***Isothermal flow*** - extremely efficient cooling (heat transport).

heat transport timescale << dynamic timescale

This implies the balance between heating and cooling, hence a constant temperature.

From the ideal gas law,

$$P = \frac{N}{V} kT = \rho \frac{kT}{m} \longrightarrow \begin{array}{l} P \propto \rho \\ P \propto V^{-1} \end{array}$$

- **In general**, we have

$$\begin{array}{l} P \propto \rho^{\gamma} \\ P \propto V^{-\gamma} \end{array} \quad (\gamma = 1 \text{ for isothermal gas})$$

A gas that has an equation of state with this power-law form is called a ***polytope***, from the Greek polytropos, meaning “turning many ways” or “versatile.”

(A polystrope should not be confused with a polytrope, which is the n-dimensional generalization of a 2D polygon and 3D polyhedron.)

- ***Specific internal energy*** of the gas (***per unit mass***):

$$\epsilon \equiv U/M \qquad U/N = \frac{f}{2}kT \longrightarrow \epsilon = \frac{f}{2}\frac{kT}{m} \quad \text{or} \quad \epsilon = \frac{1}{\gamma-1}\frac{kT}{m} = \frac{1}{\gamma-1}\frac{P}{\rho}$$

- ***Total Energy (per unit volume)***:
  - ***Internal energy per unit volume***:
  - ***Kinetic energy due to bulk motion, per unit volume***:
  - ***Work on unit volume***:

$$\mathcal{E}_{\rm int} = \rho\epsilon = \frac{1}{\gamma-1}P$$

$$\mathcal{E}_{\rm kin} = \rho\frac{u^2}{2}$$

$$\mathcal{E}_{\rm mech} = \frac{PdV}{dV} = P$$

$$\begin{aligned}\mathcal{E} &= \mathcal{E}_{\rm int} + \mathcal{E}_{\rm kin} + \mathcal{E}_{\rm mech}\\ &= \rho\left(\frac{u^2}{2} + \epsilon\right) + P\end{aligned} \longrightarrow \mathcal{E} = \rho\frac{u^2}{2} + \frac{\gamma}{\gamma-1}P$$

- ***Energy conservation***:

$$\frac{\partial \mathcal{E}}{\partial t} = -\frac{\partial (u\mathcal{E})}{\partial x}$$

$$\frac{\partial}{\partial t}\left(\rho\frac{u^2}{2} + \frac{\gamma}{\gamma-1}P\right) = -\frac{\partial}{\partial x}\left[u\left(\rho\frac{u^2}{2} + \frac{\gamma}{\gamma-1}P\right)\right]$$

Sound Wave

- Suppose that we are surrounded by an ideal gas with a plane parallel symmetry:
  - We consider a region where the gas has initially a uniform density, pressure, and no bulk velocity: $\rho_0, \; P_0, \; u_0 = 0$

    In the uniform gas, we introduce small perturbations of the form:

$$\begin{aligned} \rho(x,t) &= \rho_0 + \rho_1(x,t) \\ u(x,t) &= u_1(x,t) \\ P(x,t) &= P_0 + P_1(x,t) \end{aligned} \longrightarrow \begin{aligned} P_1 &= P - P_0 \\ &\propto (\rho_0+\rho_1)^\gamma - \rho_0^\gamma \\ &\propto \gamma \rho_0^{\gamma-1}\rho_1 \end{aligned} \longrightarrow P_1 = \frac{\gamma P_0}{\rho_0}\rho_1$$

We obtain:

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= -\frac{\partial(\rho u)}{\partial x} \\ \rho\frac{\partial u}{\partial t} &= -\rho u \frac{\partial u}{\partial x} - \frac{\partial P}{\partial x} \end{aligned} \longrightarrow \begin{aligned} \frac{\partial \rho_1}{\partial t} &= -\rho_0 \frac{\partial u_1}{\partial x} \\ \rho_0 \frac{\partial u_1}{\partial t} &= -\frac{\partial P_1}{\partial x} = -\frac{\gamma P_0}{\rho_0}\frac{\partial \rho_1}{\partial x} \end{aligned} \longrightarrow \boxed{\frac{\partial^2 \rho_1}{\partial t^2} = -\frac{\gamma P_0}{\rho_0}\frac{\partial^2 \rho_1}{\partial x^2}}$$

  - The resulting equation represents a **sound wave (acoustic wave)** with a constant sound speed:

$$\boxed{c_s = \left(\frac{\gamma P}{\rho}\right)^{1/2} = \left(\frac{\gamma k T}{m}\right)^{1/2}} \qquad \boxed{c_s \propto \rho^{(\gamma-1)/2}}$$

For $\gamma > 1$ sound travels more rapidly in a denser gas.

- The sound speed is of the same order as the mean thermal velocity:

$$c_s = 1.2\,\mathrm{km\,s^{-1}} \left(\frac{\gamma}{5/3}\right)^{1/2} \left(\frac{m}{m_p}\right)^{-1/2} \left(\frac{T}{100\,\mathrm{K}}\right)^{1/2}$$

$$(m_p = \text{proton mass})$$

- ***Sound crossing time***:
  - sound crossing time = time it takes for a signal to cross a region of size *L*:

$$t_{\text{cross}} = L/c_s$$

  - A small pressure gradient tends to be smoothed out within the sound crossing time. Generally, when a stationary gas is disturbed, the resultant changes in velocity, density, pressure, and temperature are communicated downstream at the sound speed.

    Fast changes occurring on timescales $\ll t_{\text{cross}}$ *will survive, and a shock front forms.*

    Slow changes occurring on timescales $\gg t_{\text{cross}}$ will be damped.

- ***Mach number*** = gas velocity / sound speed

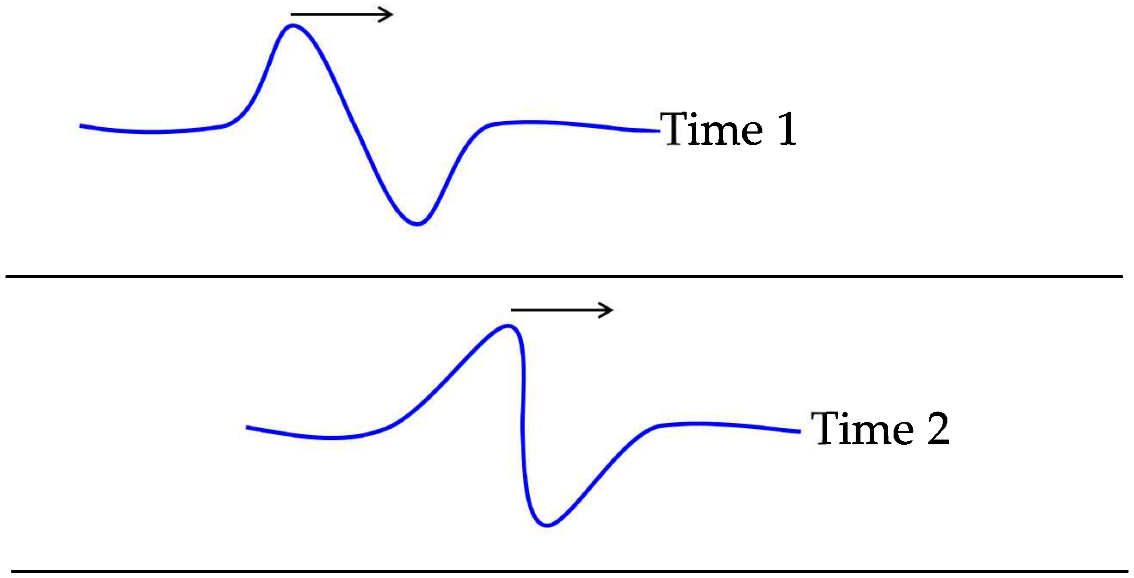
$$\boxed{\mathcal{M} \equiv u/c_s}$$

$\mathcal{M} > 1$ supersonic
$\mathcal{M} < 1$ subsonic

Shock

- Shock
  - A low-amplitude sound wave traveling through a medium will be adiabatic; that is it will not increase the entropy of the gas through which it passes.
  - For an adiabatic process, the equation of state for the gas is

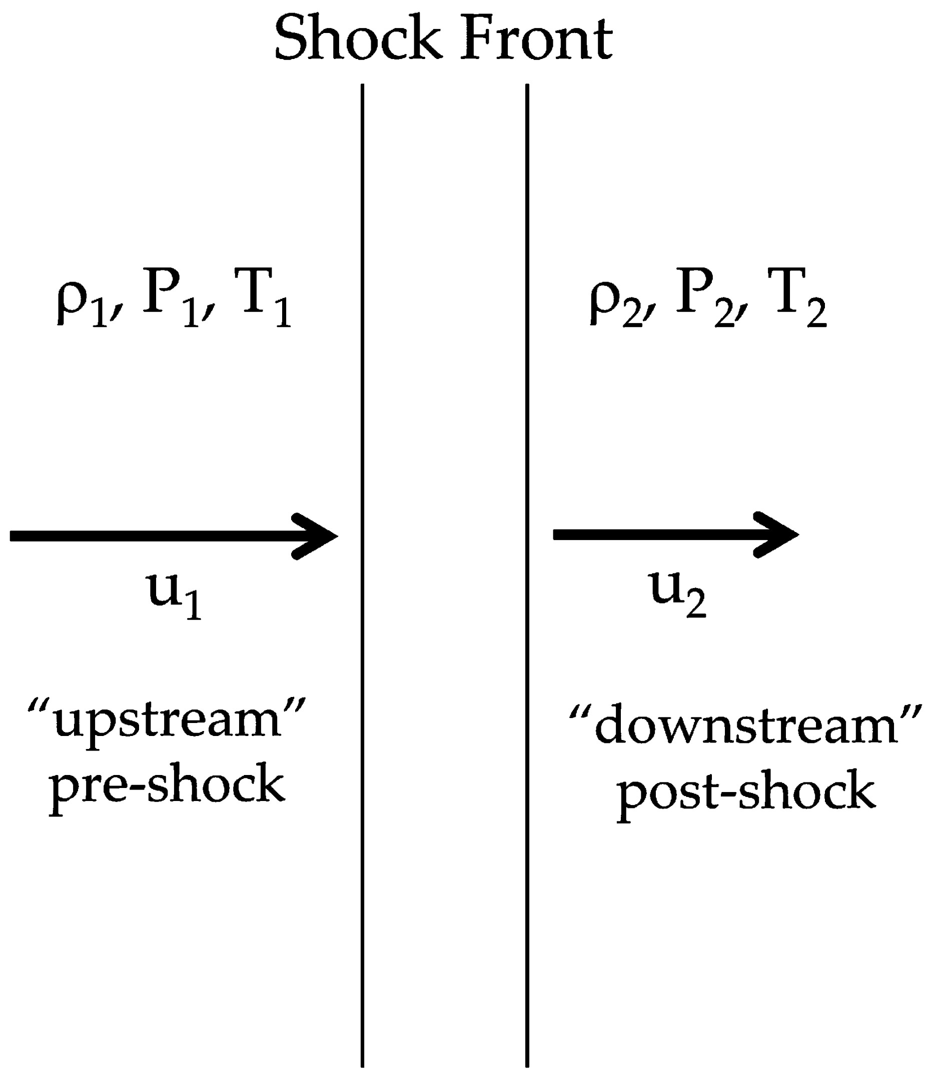
$$c_s \propto \rho^{(\gamma-1)/2}$$



  - Thus, for $\gamma > 1$, sound travels more rapidly in a denser gas.
  - ***For a supersonic gas, the motion itself is faster than the speed of communication, and instead of a smooth transition, the physical quantities (density, pressure, and temperature) undergo a sudden change in values over a small distance.*** This phenomenon is referred to as a ***shock***.
  - We define the shock front as the region over which the velocity, density, and pressure of the gas undergo sudden changes. The shock front is a layer whose thickness is comparable to the mean free path between particle collisions.
  - The ordinary sound that we hear every day will not, in practice, steepen into shocks.
  - However, high amplitude pressure fluctuations will rapidly steepen into shocks.

Shock Front

- Jump condition (***Rankine-Hugoniot conditions***)
  - Let
    $$\rho = \text{mass density}, \; T = \text{temperature},$$
    $$m = \text{mean molecular mass}$$
  - If a patch is small compared to the shock front's radius of curvature, then we can treat the shock front as if it has ***plane parallel*** symmetry.
  - ***It is convenient to use a frame of reference in which the shock front is stationary.***
  - Let us consider a shock propagating with velocity V_s into a gas that is previously at rest. In the frame of reference of the shock, the gas in the pre-shock region is approaching at a velocity of $-V_s$.
  - In this frame, the bulk velocity $u_1 = -V_s$ of the pre-shock (upstream) gas toward the shock front. The bulk velocity u_2 of the post-shock (downstream) gas points away from the shock front.



Plane parallel steady-state shock, in the reference frame of the shock front.

- Let’s consider a steady state solution.
    - The gas properties immediately before being shocked (“1”) and immediately after being shocked (“2”) are obtained from the conservation laws:

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2$$

$$u_1 \left(\rho_1 \frac{u_1^2}{2} + \frac{\gamma}{\gamma - 1} P_1 \right) = u_2 \left(\rho_2 \frac{u_2^2}{2} + \frac{\gamma}{\gamma - 1} P_2 \right)$$

Dividing the third equation with the first equation:

$$\frac{u_1^2}{2} + \frac{\gamma}{\gamma - 1} \frac{P_1}{\rho_1} = \frac{u_2^2}{2} + \frac{\gamma}{\gamma - 1} \frac{P_2}{\rho_2}$$

In summary,

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2$$

$$\frac{u_1^2}{2} + \frac{\gamma}{\gamma - 1} \frac{P_1}{\rho_1} = \frac{u_2^2}{2} + \frac{\gamma}{\gamma - 1} \frac{P_2}{\rho_2}$$

Here, *we assume that an adiabatic index is the same on both sides of the shock front.*

- From the three equations, we should be able to derive the changes, ρ_2/ρ_1, u_2/u_1, and P_2/P_1 across the shock.

  It is convenient to use a dimensionless number, the Mach number of the upstream:

$$\mathcal{M}_1 = u_1/c_1,\ c_1^2 = \frac{\gamma P_1}{\rho_1} \quad \rightarrow \quad P_1 = \frac{u_1^2\rho_1}{\gamma\mathcal{M}_1^2}$$

(1) To find the equation for densities:

$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2$$

$$\rho_1 u_1 = \rho_2 u_2 \text{ and } P_1 = \frac{u_1^2\rho_1}{\gamma\mathcal{M}_1^2} \qquad \rightarrow \quad \rho_1 u_1^2 + u_1^2\frac{\rho_1}{\gamma\mathcal{M}_1^2} = \frac{(\rho_1 u_1)^2}{\rho_2} + P_2$$

$$\rightarrow \quad P_2 = \rho_1 u_1^2\left(1 + \frac{1}{\gamma\mathcal{M}_1^2} - \frac{\rho_1}{\rho_2}\right)$$

Inserting these relations into the energy conservation equation:

$$\frac{u_1^2}{2} + \frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1} = \frac{u_2^2}{2} + \frac{\gamma}{\gamma-1}\frac{P_2}{\rho_2}$$

$$\rightarrow \quad \frac{u_1^2}{2} + \frac{1}{\gamma-1}\frac{u_1^2}{\mathcal{M}_1^2} = \frac{1}{2}\left(\frac{\rho_1 u_1}{\rho_2}\right)^2 + \frac{\gamma}{\gamma-1}\frac{\rho_1 u_1^2}{\rho_2}\left(1 + \frac{1}{\gamma\mathcal{M}_1^2} - \frac{\rho_1}{\rho_2}\right)$$

$$\rightarrow \quad \frac{1}{2} + \frac{1}{\gamma-1}\frac{1}{\mathcal{M}_1^2} = \frac{1}{2}\left(\frac{\rho_1}{\rho_2}\right)^2 + \frac{\gamma}{\gamma-1}\frac{\rho_1}{\rho_2}\left(1 + \frac{1}{\gamma\mathcal{M}_1^2} - \frac{\rho_1}{\rho_2}\right)$$

$$ax^2 + bx - c = 0$$

where $x = \dfrac{\rho_1}{\rho_2}$

$$a = \frac{1}{2} - \frac{\gamma}{\gamma-1}$$

$$b = \frac{\gamma}{\gamma-1} + \frac{1}{(\gamma-1)\mathcal{M}_1^2}$$

$$c = \frac{1}{2} + \frac{1}{(\gamma-1)\mathcal{M}_1^2}$$

$$x = \frac{b^2 \pm \sqrt{b^2 + 4ac}}{2a}$$

$$\begin{aligned}\frac{\rho_1}{\rho_2} &= \frac{-\left[\frac{\gamma}{\gamma-1} + \frac{1}{(\gamma-1)\mathcal{M}_0^2}\right] \pm \frac{\mathcal{M}_1^2-1}{\mathcal{M}_1^2(\gamma-1)}}{1 - \frac{2\gamma}{\gamma-1}} \\ &= \frac{\gamma\mathcal{M}_1^2 + 1 \mp (\mathcal{M}_1^2 - 1)}{(\gamma+1)\mathcal{M}_1^2}\end{aligned}$$

$$\longrightarrow$$

$$\frac{\rho_1}{\rho_2} = 1 \quad \text{or} \quad \frac{(\gamma-1)\mathcal{M}_1^2 + 2}{(\gamma+1)\mathcal{M}_1^2}$$

$$\therefore \ \frac{\rho_2}{\rho_1} = \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2 + 2}$$

(2) Now, we obtain the equation for pressures:

Divide the following equation

$$P_2 = \rho_1 u_1^2 \left(1 + \frac{1}{\gamma\mathcal{M}_1^2} - \frac{\rho_1}{\rho_2}\right)$$

with this

$$P_1 = \frac{u_1^2 \rho_1}{\gamma\mathcal{M}_1^2}$$

$$\longrightarrow$$

$$\begin{aligned}\frac{P_2}{P_1} &= \gamma\mathcal{M}_1^2\left(1 + \frac{1}{\gamma\mathcal{M}_1^2} - \frac{\rho_1}{\rho_2}\right) \\ &= \gamma\mathcal{M}_1^2\left(1 + \frac{1}{\gamma\mathcal{M}_1^2} - \frac{(\gamma-1)\mathcal{M}_1^2 + 2}{(\gamma+1)\mathcal{M}_1^2}\right)\end{aligned}$$

$$\therefore \ \frac{P_2}{P_1} = \frac{2\gamma\mathcal{M}_1^2 - (\gamma-1)}{\gamma+1}$$

(3) Using the ideal gas law:

$$P = \frac{\rho k T}{m} \quad \longrightarrow \quad \frac{T_2}{T_1} = \frac{\rho_1}{\rho_2}\frac{P_2}{P_1}$$

Using the equations for densities and pressures:

$$\therefore \ \frac{T_2}{T_1} = \frac{\left[(\gamma-1)\mathcal{M}_1^2 + 2\right]\left[2\gamma\mathcal{M}_1^2 - (\gamma-1)\right]}{(\gamma+1)^2\mathcal{M}_1^2}$$

In summary, we obtain the jump conditions:

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2+2} = \frac{u_1}{u_2}$$

$$\frac{P_2}{P_1} = \frac{2\gamma\mathcal{M}_1^2-(\gamma-1)}{\gamma+1}$$

$$\frac{T_2}{T_1} = \frac{\left[(\gamma-1)\mathcal{M}_1^2+2\right]\left[2\gamma\mathcal{M}_1^2-(\gamma-1)\right]}{(\gamma+1)^2\mathcal{M}_1^2}$$

In the ***lab frame***, let V_s = shock velocity, $v_1,\ v_2$ = gas velocities in upstream (pre-shock) and downstream (post-shock), respectively $(v_1 = 0)$.

Using $u_1 = -V_s$ and $u_2 = v_2 - V_s$, we have

$$\frac{-V_s}{v_2 - V_s} = \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2+2}$$

Downstream velocity in the lab frame:

$$v_2 = \frac{2(\mathcal{M}_1^2-1)}{(\gamma+1)\mathcal{M}_1^2} V_s$$

Note a typo in Equation (16.12) of Kwok's book.

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2+2} = \frac{u_1}{u_2}$$

$$\frac{P_2}{P_1} = \frac{2\gamma\mathcal{M}_1^2-(\gamma-1)}{\gamma+1}$$

$$\frac{T_2}{T_1} = \frac{\left[(\gamma-1)\mathcal{M}_1^2+2\right]\left[2\gamma\mathcal{M}_1^2-(\gamma-1)\right]}{(\gamma+1)^2\mathcal{M}_1^2}$$

For a strong shock: $\mathcal{M}_1 \gg 1$

$$P_2 \approx \frac{2\gamma\mathcal{M}_1^2}{\gamma+1}P_1 \xrightarrow{P_1=c_1^2\frac{\rho_1}{\gamma}} \frac{2\gamma(u_1/c_1)^2}{\gamma+1} \quad c_1^2\frac{\rho_1}{\gamma}$$

$$T_2 \approx \frac{2(\gamma-1)\gamma}{(\gamma+1)^2}\mathcal{M}_1^2 T_1 = \frac{2(\gamma-1)\gamma}{(\gamma+1)^2}\left(\frac{u_1}{c_1}\right)^2 T_1$$

$$\frac{\rho_2}{\rho_1} \simeq \frac{\gamma+1}{\gamma-1}$$

$$\frac{u_2}{u_1} \simeq \frac{\gamma-1}{\gamma+1}$$

$$P_2 \simeq \frac{2}{\gamma+1}\rho_1 u_1^2$$

$$T_2 \simeq \frac{2(\gamma-1)}{(\gamma+1)^2}\frac{m}{k}u_1^2$$

monatomic gas: $\gamma = 5/3$

$$\frac{\rho_2}{\rho_1} \simeq 4$$

$$\frac{u_2}{u_1} \simeq \frac{1}{4}$$

$$P_2 \simeq \frac{3}{4}\rho_1 u_1^2$$

$$T_2 \simeq \frac{3}{16}\frac{m}{k}u_1^2$$

speed of the downstream in the laboratory frame:

$$v_2 \simeq \frac{2}{(\gamma+1)}V_s$$

$$v_2 \simeq \frac{3}{4}V_s$$

For an isothermal shock: $\gamma = 1$

$$\frac{\rho_2}{\rho_1} = \mathcal{M}_1^2 = \frac{u_1}{u_2}$$

$$P_2 = \mathcal{M}_1^2 P_1 = \rho_1 u_1^2$$

$$T_2 = T_1$$

$$\longrightarrow \quad u_1 u_2 = c_1^2 \qquad c_2 = c_1$$

speed of the downstream in the laboratory frame:

$$v_2 = \left(1 - \frac{1}{\mathcal{M}_1^2}\right)V_s$$

- Consider a strong shock
  - ***No matter how strong the shock is, the gas can only be compressed by a factor of at most 4***:

$$\frac{\rho_2}{\rho_1} \approx 4 \qquad \text{for } \gamma = 5/3$$

(monatomic gas)

$$P_2 \approx \frac{3}{4}\rho_1 u_1^2$$

$$T_2 \approx \frac{3}{16}\frac{m}{k}u_1^2$$

Note that the mean molecular mass (mass per particle) is

$$m = \frac{1.4 m_\mathrm{H}}{1.1} = 1.273 m_\mathrm{H} \quad \text{for neutral gas}$$

$$m = \frac{1.4 m_\mathrm{H}}{2.3} = 0.609 m_\mathrm{H} \quad \text{for ionized gas}$$

$$n \simeq 2.3 n_\mathrm{H}$$

for ionized gas,
one electron from an ionized hydrogen
two electrons from a doubly-ionized helium.

  - In the lab frame, V_s = shock velocity, $v_1,\ v_2$ = gas velocities in upstream and downstream, respectively.

$$u_1 = v_1 - V_s = -V_s \ (v_1 = 0)$$

$$u_2 = v_2 - V_s$$

  - Then, the post-shock velocity is

$$\frac{u_2}{u_1} = \frac{v_s - V_\mathrm{s}}{-V_\mathrm{s}} = \frac{1}{4} \quad \Rightarrow \quad v_2 = \frac{3}{4}V_\mathrm{s}$$

  - Hence, ***the post-shock moves in the same direction as the shock front with a velocity of 3/4 of the shock velocity.***

- Then, the post-shock pressure, temperature, specific internal energy, and specific kinetic energy are, respectively,

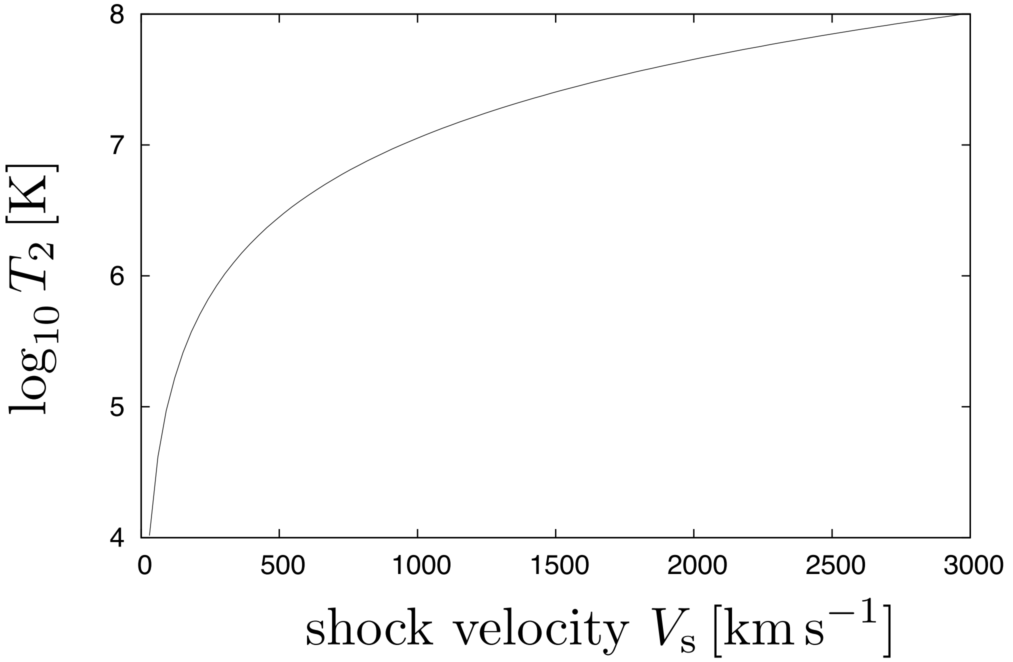
$$P_2 = \frac{3}{4}\rho_1 V_{\rm s}^2$$
$$T_2 = \frac{3m}{16k}V_{\rm s}^2$$

for $\gamma = 5/3$

$$\frac{\mathcal{E}_{\rm int,2}}{\rho_2} = \epsilon_2 = \frac{3}{2}\frac{P_2}{\rho_2} = \frac{3}{2}\frac{(3/4)\rho_1 V_{\rm s}^2}{4\rho_1}$$
$$\frac{\mathcal{E}_{\rm kin,2}}{\rho_2} = \frac{1}{2}v_2^2$$

$$\frac{\mathcal{E}_{\rm int,2}}{\rho_2} = \frac{9}{32}V_{\rm s}^2$$
$$\frac{\mathcal{E}_{\rm kin,2}}{\rho_2} = \frac{9}{32}V_{\rm s}^2$$

Post-shock temperature



- A strong shock can produce very high pressures and temperatures. An interstellar shock front with propagation speed $V_s \sim 1000$ km s^{-1} (typical for a supernova shock wave) produces shock heated gas with

$$T_2 \approx 1.38\times10^7\,{\rm K}\left(\frac{m}{0.609m_{\rm H}}\right)\left(\frac{V_s}{1000\,{\rm km\,s^{-1}}}\right)^2$$ or $$T_2 \approx 1.38\times10^5\,{\rm K}\left(\frac{m}{0.609m_{\rm H}}\right)\left(\frac{V_s}{100\,{\rm km\,s^{-1}}}\right)^2$$

assuming the shocks gas is fully ionized hydrogen.

- In general, shock fronts convert supersonic gas into subsonic gas in the shock's frame of reference. Shocks increase density, pressure, and temperature, and decrease bulk velocity relative to the shock front. *Shocks act as entropy generators.*

Hot Ionized Medium

- Hot Gas Cooling
- Supernova Remnant
- Local Hot Bubble

General Properties of the HIM

- Hot Ionized Medium, coronal gas
  - About half the volume of the ISM in our Galaxy is occupied by the HIM.
  - Temperature ~ 10^6 K.
  - Typical ion number density n ~ 0.004 cm^{-3}
  - It provides only ~ 0.2% of the mass of the ISM, despite being the largest contributor to its volume.
  - The HIM is hot because it has been heated by shock fronts that result from supernova explosions.
  - ***We live in the "Local Bubble", which is ~ 100 pc in size. The Local Bubble is thought to have been blown by a supernova that went off ~ 10 Myr ago.***

Collisional Ionization Equilibrium

- CIE
  - CIE assumes that the plasma is in a steady state, and that collisional ionization, charge exchange, radiative recombination, and dielectronic recombination are the only processes altering the ionization balance.
    - Note that the reverse process to collisional ionization is a three-body recombination, which is unlikely to occur.
  - The ionization fractions for each element depend only on the gas temperature, with no dependence on the gas density.
- Ionization fraction
  - For hydrogen, the balance equation is : ionization rate = recombination rate

$$n_e n(\mathrm{H}^0) k_{\mathrm{ci,H}} = n_e n(\mathrm{H}^+) \alpha_{\mathrm{A,H}} \qquad\qquad n(\mathrm{H}^0) + n(\mathrm{H}^+) = n(\mathrm{H})$$

  - The rate coefficients for collisional ionization and radiative recombination are:

$$\begin{aligned} k_{\mathrm{ci,H}} &= 5.849 \times 10^{-9}\, T_4^{1/2} e^{-15.782/T_4}\, [\mathrm{cm^3\,s^{-1}}] \\ \alpha_{\mathrm{A,H}} &= 4.13 \times 10^{-13}\, T_4^{-0.7131-0.0115\ln T_4}\, [\mathrm{cm^3\,s^{-1}}] \quad \text{for} \quad 30\,\mathrm{K} < T < 3\times 10^4\,\mathrm{K} \\ &= 5 \times 10^{-16} T_7^{-1.5} \quad \text{for} \quad T > 10^6\,\mathrm{K} \end{aligned}$$

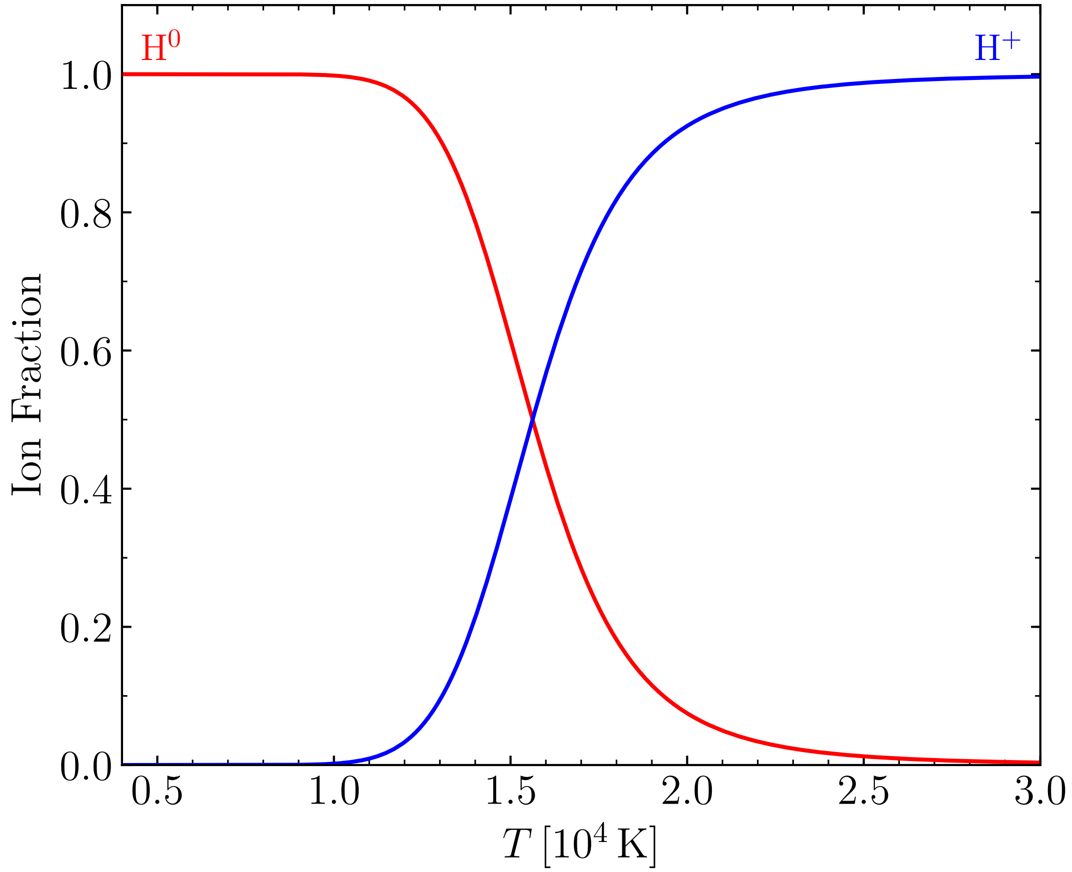
[from Draine]

$$\alpha_{\mathrm{A,H}} = 1.269 \times 10^{-13}\, [\mathrm{cm^3\,s^{-1}}]\, \frac{x^{1.503}}{(1 + (x/0.522)^{0.47})^{1.923}} \quad \text{where } x = 2 \times 157807\,\mathrm{K}/T$$

[Hui & Gendin 1997, MNRAS]

- The ionization fraction is

$$x \equiv \frac{n(\mathrm{H}^+)}{n(\mathrm{H}^0) + n(\mathrm{H}^+)} = \frac{k_{\mathrm{ci,H}}}{k_{\mathrm{ci,H}} + \alpha_{\mathrm{A,H}}}$$



  - The ion fractions are

$$x \approx 0.002 \quad \text{at} \quad T = 10^4\,\mathrm{K}$$
$$1 - x \approx 3 \times 10^{-7} \quad \text{at} \quad T = 10^6\,\mathrm{K}$$

H II regions with $T = 10^4$ K are photoionized by UV photons from hot stars.

Hydrogen gas with $T = 10^6$ K is almost entirely collisionally ionized.

- For Helium, the balance equations are:

$$n(\mathrm{He}^+)\alpha_{10} = n(\mathrm{He}^0)k_{01}$$
$$n(\mathrm{He}^+)k_{12} = n(\mathrm{He}^{2+})\alpha_{21}$$
$$n(\mathrm{He}) = n(\mathrm{He}^0) + n(\mathrm{He}^+) + n(\mathrm{He}^{2+})$$

Here, ij indicates $X^{i+} \rightarrow X^{j+}$.

- The rate coefficients are

from Cen (1992, ApJS)

$$k_{01} = 2.39 \times 10^{-11}\, T^{1/2} e^{-285,335/T}$$

$$k_{12} = 5.68 \times 10^{-12}\, T^{1/2} e^{-631,515/T}$$

$$\alpha_{10} = 1.50 \times 10^{-10}\, T^{-0.6353}$$ radiative recombination

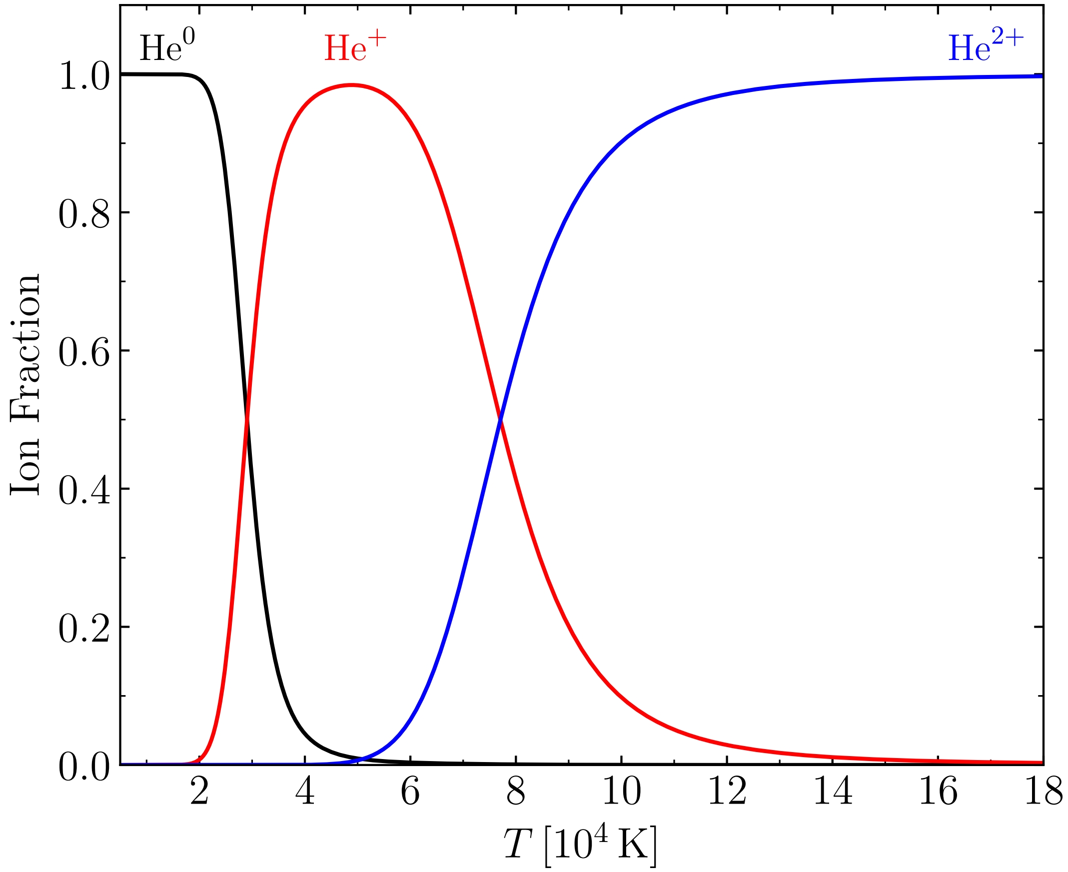
$$+ 1.9 \times 10^{-3} T^{-1.5} e^{-470,000/T} (1 + 0.3 e^{-94,000/T})$$ dielectronic recombination (but not significant)

$$\alpha_{21} = 3.36 \times 10^{-10}\, T^{-1/2} T_3^{-0.2} / (1 + T_6^{0.7})$$

- Using the above rate coefficients, the ionization fractions can be estimated as follows:

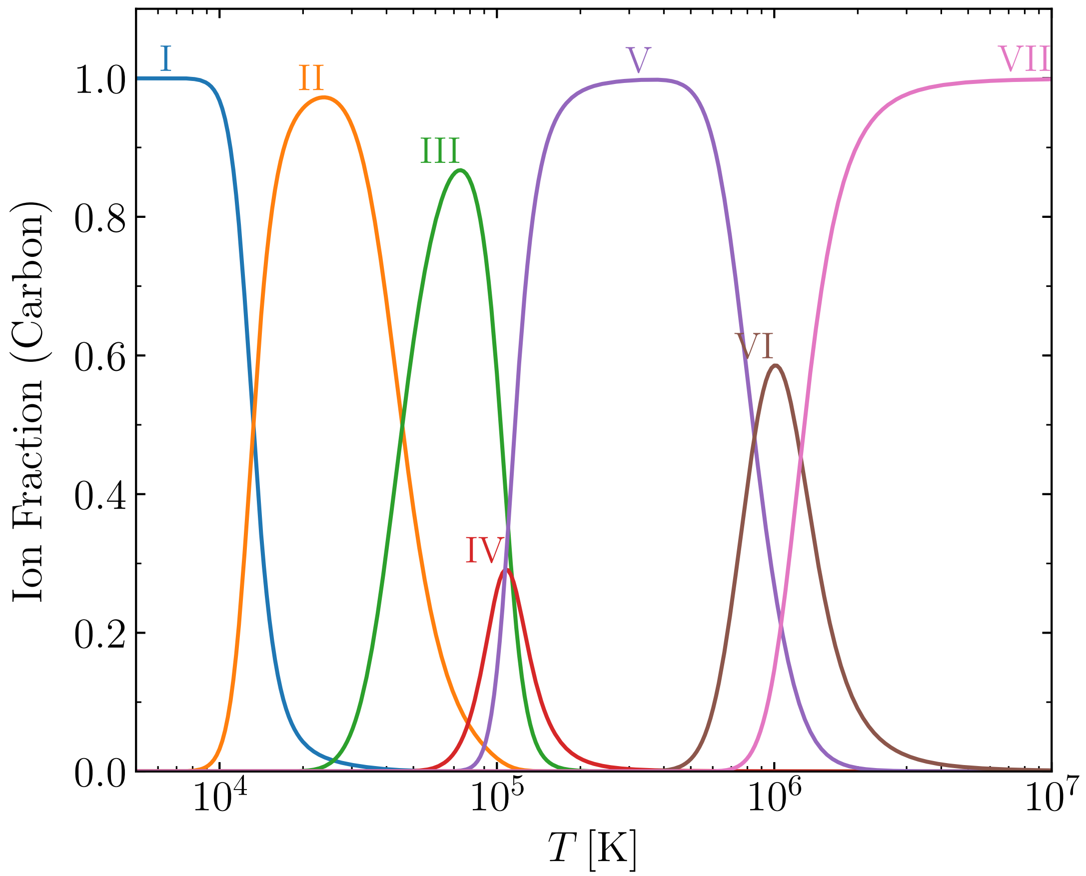
$$x \equiv \frac{n(\mathrm{He}^+)}{n(\mathrm{He})} = \frac{1}{1 + \alpha_{10}/k_{01} + k_{12}/\alpha_{21}}$$

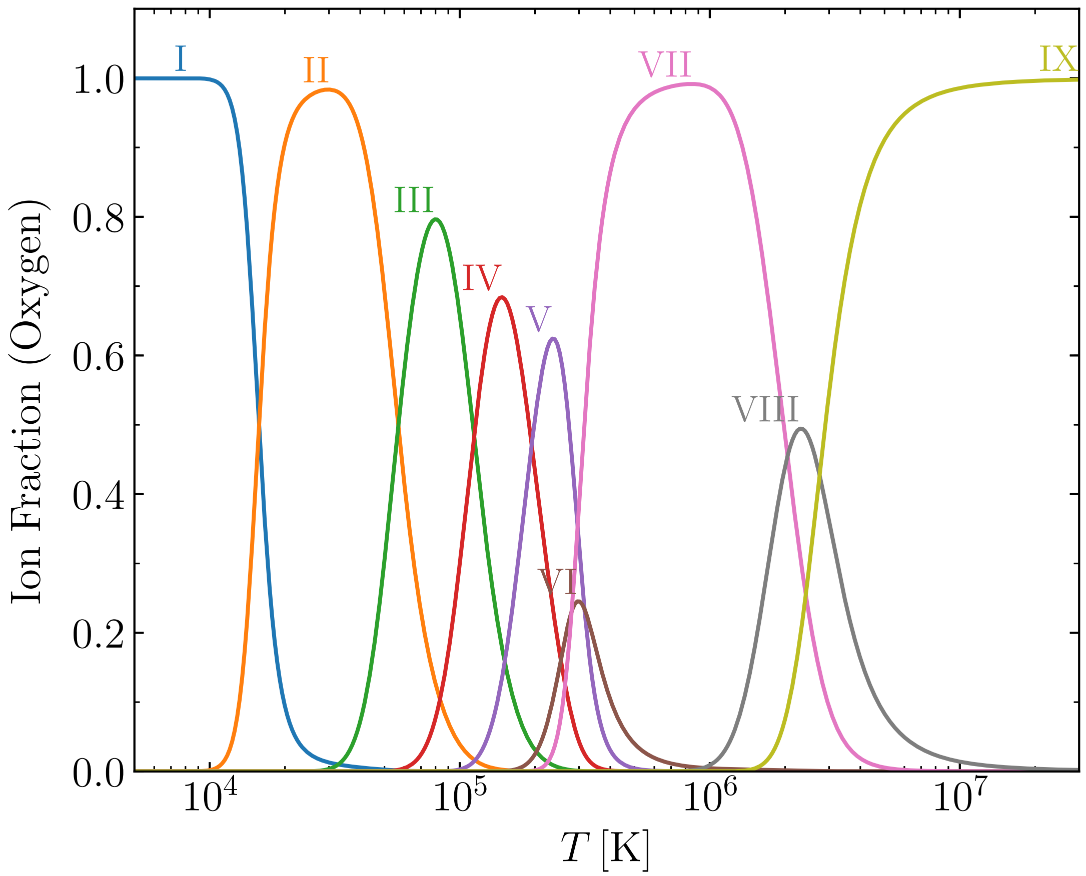
$$y \equiv \frac{n(\mathrm{He}^{2+})}{n(\mathrm{He})} = \frac{k_{12}}{\alpha_{21}} x$$



- Heavy Elements
  - The calculation is usually done numerically, for instance, using CHIANTI

    CHIANTI: https://www.chiantidatabase.org/
  - For instance, the ion fractions of Carbon and Oxygen as a function of temperature are:





- ◉ At $T \sim 10^6\,\mathrm{K}$, we expect a mix of C V, C VI, and C VII.
- ◉ At $T \sim 4 \times 10^6\,\mathrm{K}$ and higher, almost all the carbon will be in the form of fully ionized C VII.

The figures were calculated using CHIANTI.

Cooling in CIE

- ***Cooling function***
  - At temperature $T > 10^4\,\mathrm{K}$, ionization of hydrogen provides enough free electrons so that collisional excitation of atoms or ions is dominated by electron collisions.
  - At low densities, every collisional excitation is followed by a radiative decay, and ***the rate of removal of thermal energy per unit volume*** can be written:

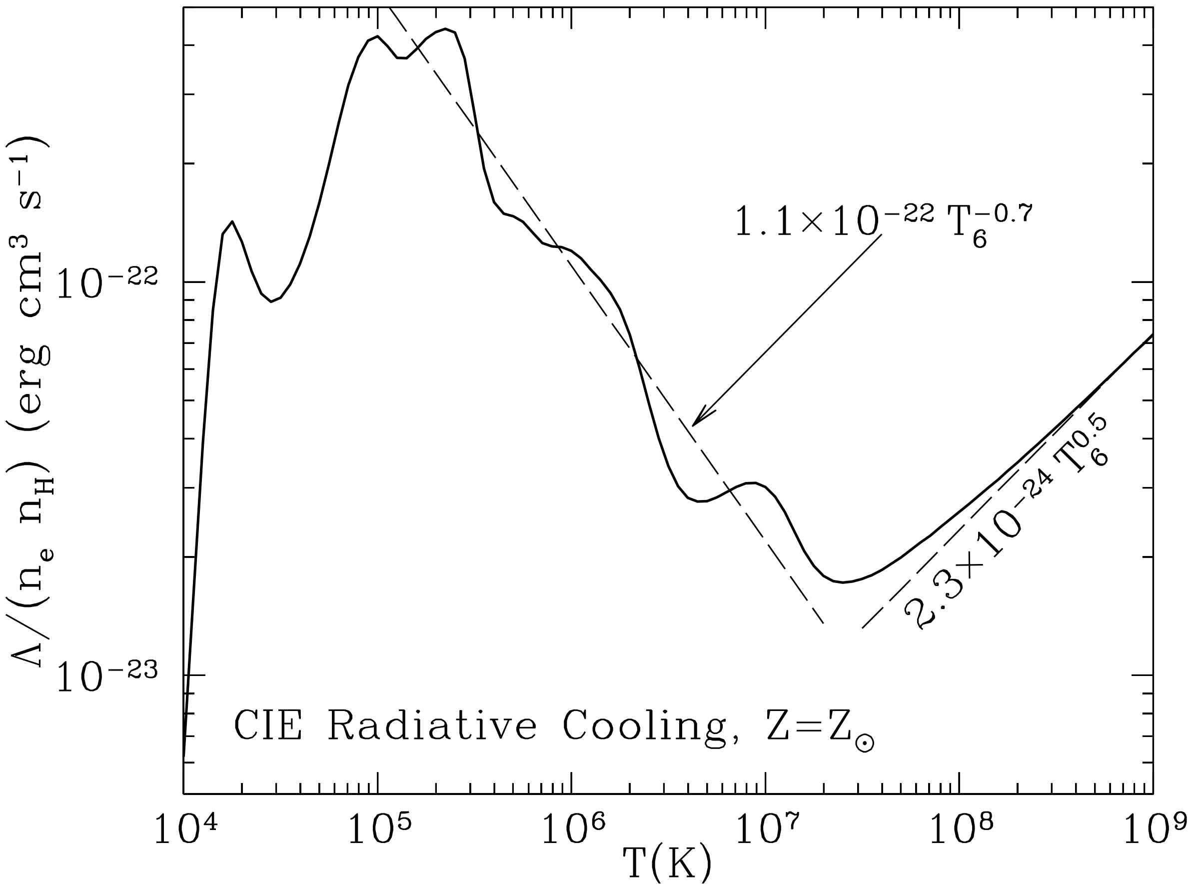
$$\boxed{\Lambda = n_e n_{\mathrm{H}} f_{\mathrm{cool}}(T)}$$

The ***radiative cooling function*** $f_{\mathrm{cool}}(T) \equiv \Lambda / n_{\mathrm{H}} n_e$ is a function of ***temperature*** and of the ***elemental abundances*** relative to hydrogen.

- *At high densities*, radiative cooling can be suppressed by collisional deexcitation, and the cooling function will then depend on density, in addition to T and elemental abundances.

- *If ionizing radiation is present*, the ionization balance may depart from CIE, and the radiative cooling function will also depend on the spectrum and intensity of the ionizing radiation.

Fig 34.1
Draine



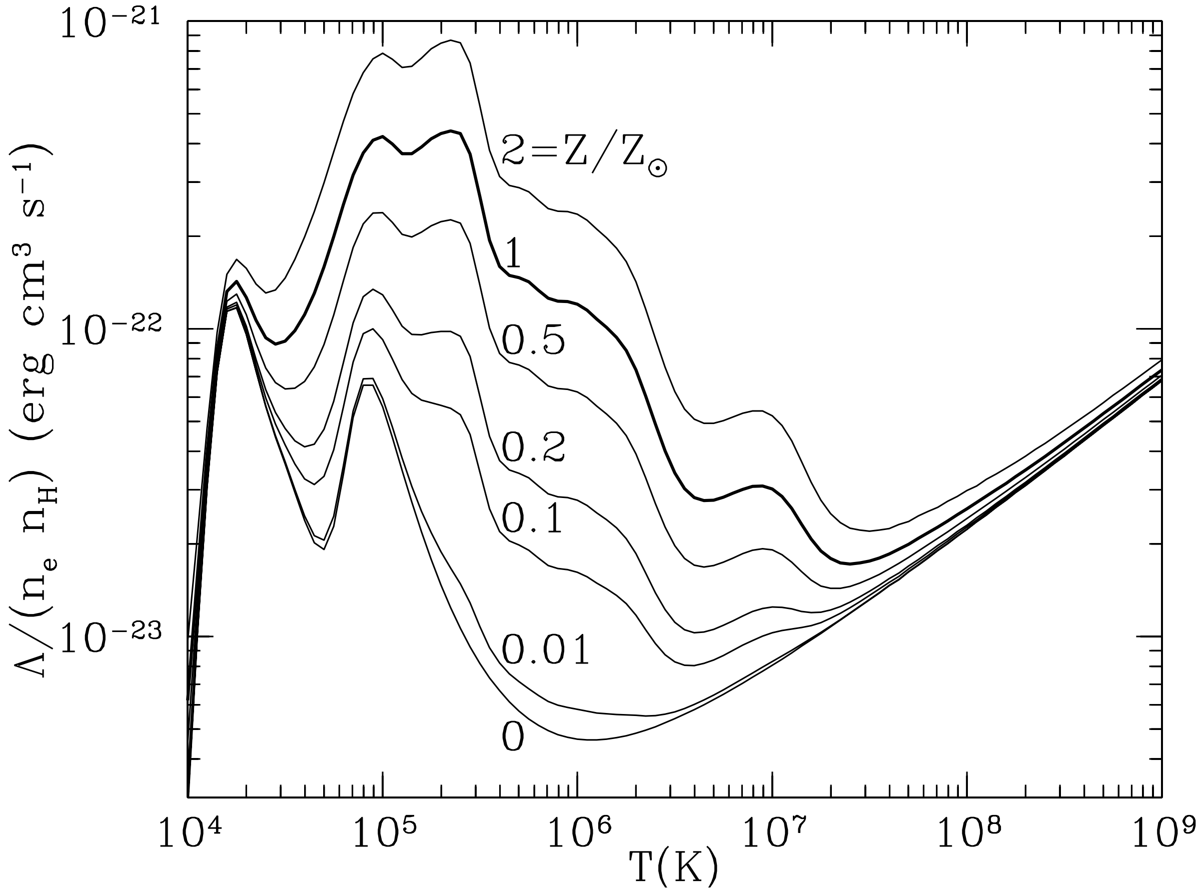
Radiative Cooling Function for solar-abundance

- At $T < 10^7\,\mathrm{K}$, the cooling is dominated by collisional excitation of bound electrons.
- At high temperatures, the ions are fully stripped of electrons, and bremsstrahlung (free-free) cooling dominates.

$$\Lambda/n_e n_{\mathrm{H}} \approx 1.1 \times 10^{-22}\, T_6^{-0.7}\, [\mathrm{erg\,cm^3\,s^{-1}}]$$
$$(10^5 < T < 10^{7.3}\,\mathrm{K})$$

$$\Lambda/n_e n_{\mathrm{H}} \approx 2.3 \times 10^{-24}\, T_6^{0.5}\, [\mathrm{erg\,cm^3\,s^{-1}}]$$
$$(T > 10^{7.3}\,\mathrm{K})$$

Fig 34.2
Draine

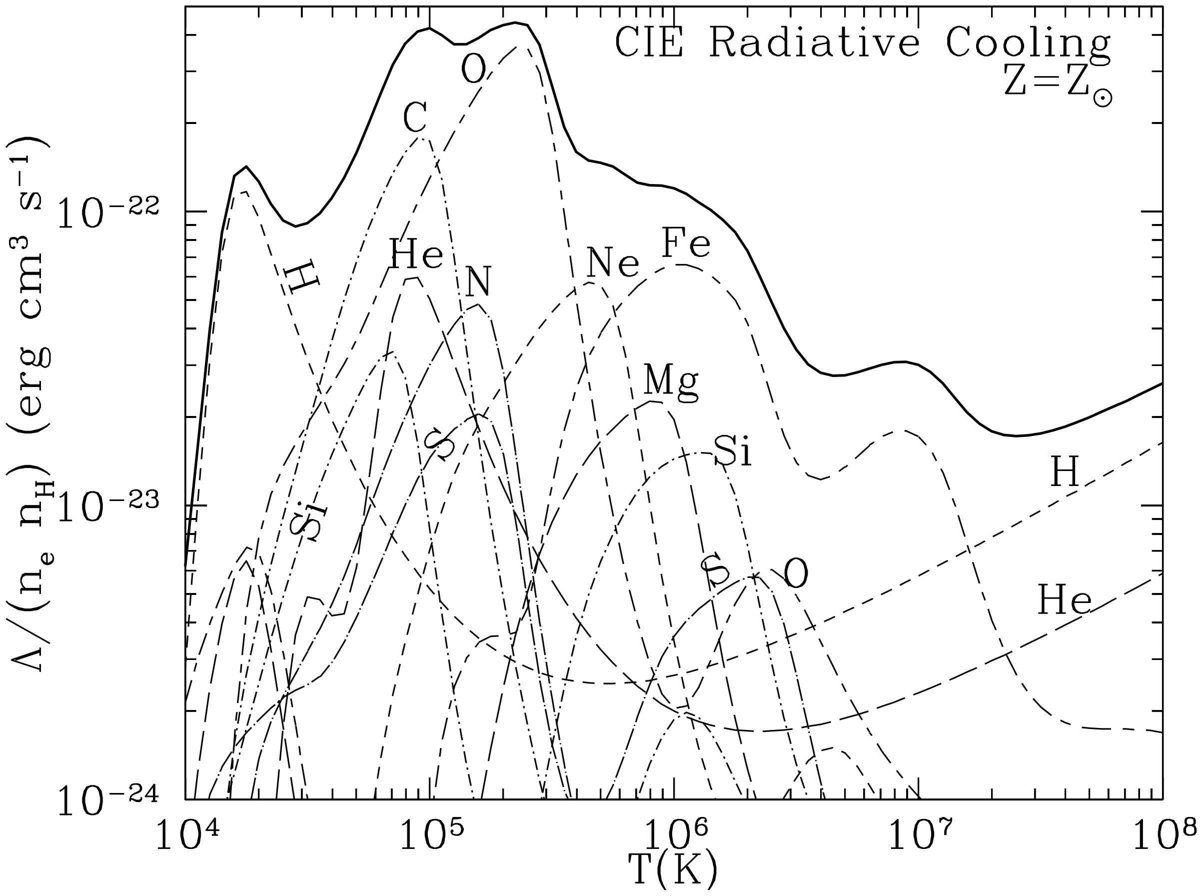


Cooling Function for different abundances

- In most applications, the abundances of elements beyond He can be assumed to be scale up and down together.

The cooling functions were calculated using CHIANTI.

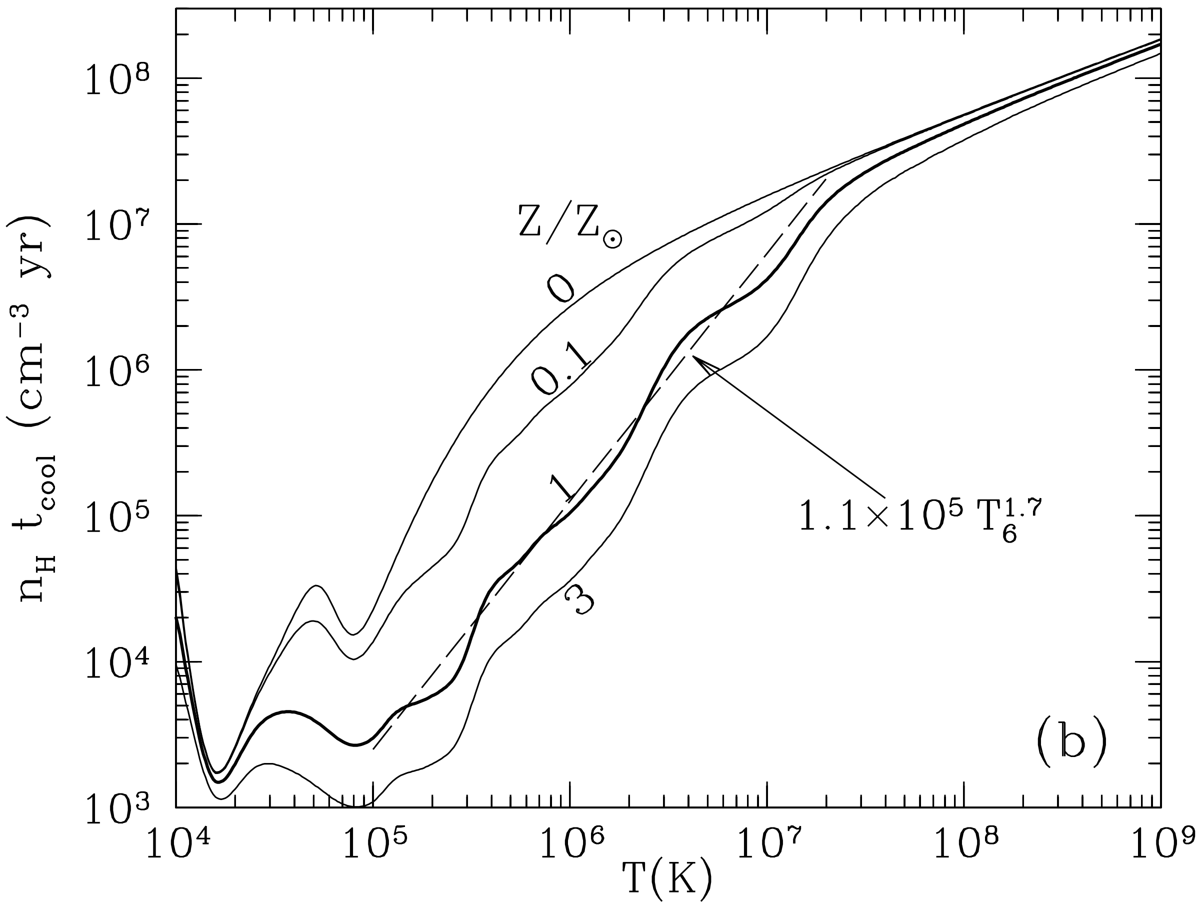
Fig 34.3
Draine



Radiative Cooling Function, with contributions from selected elements. In this calculation, the solar abundance is assumed.

- At $10^4\,\mathrm{K} < T < 3\times10^4\,\mathrm{K}$, cooling occurs mainly by Lyα emission from collisionally excited H atoms.
- At $3\times10^4\,\mathrm{K} < T < 2\times10^7\,\mathrm{K}$, cooling occurs mainly by permitted UV lines from collisionally excited heavy ions.
- For $10^{5.8}\,\mathrm{K} < T < 10^{7.2}\,\mathrm{K}$, the cooling is dominated by Mg, Si, and Fe - elements that in cold gas are normally depleted by factors of 5 or more.

Fig 34.4
Draine



Cooling time
(using the formula given in next slides)

$$t_{\rm cool} \approx 1.1\times10^5\,T_6^{1.7}\,(n_{\rm H}/{\rm cm}^{-3})^{-1}\,[{\rm yr}]$$
$$(10^5 \lesssim T \lesssim 10^{7.3}\,{\rm K})$$

for isochoric cooling (constant density)

Cooling Time Scale [isobaric / isochoric]

- Cooling Time scale for two important cases:
  - The **first law of thermodynamics** states that:

Heat added in a system: $\boxed{dQ = dU + PdV}$

  - Using a heating and cooling rate per volume Γ and Λ , the change in heat is

$$dQ = (\Gamma - \Lambda)V dt$$

  - The change in the internal energy is

$$dU = (\Gamma - \Lambda)V dt - PdV$$

  - When there is no external heating ($\Gamma = 0$), the equation for an ideal gas with a degree of freedom f becomes:

$$\begin{array}{l} U = \frac{f}{2}NkT \\ PV = NkT \end{array} \longrightarrow \boxed{d\left(\frac{f}{2}NkT\right) = -\Lambda V dt - PdV}$$

- Consider the case of constant pressure or constant volume:

$$PdV = d(PV) - VdP = d(NkT)$$ for ***isobaric cooling (constant pressure)***

$$PdV = 0$$ for ***isochoric cooling (constant density or volume)***

Therefore,

$$\frac{d}{dt}\left(\frac{f+2}{2}NkT\right) = -\Lambda V$$ for ***isobaric cooling (constant pressure)***

$$\frac{d}{dt}\left(\frac{f}{2}NkT\right) = -\Lambda V$$ for ***isochoric cooling (constant density or volume)***

The cooling time scale are then:

$$t_{\rm cool} \equiv \frac{T}{|dT/dt|} \quad \Rightarrow \quad t_{\rm cool} = \frac{f+2}{2}\frac{nkT}{\Lambda}$$ for isobaric cooling

$$n \equiv N/V \qquad = \frac{f}{2}\frac{nkT}{\Lambda}$$ for isochoric cooling

Here, the number density includes all particles (molecules, atoms, ions, electrons)

Time Scales in the HIM

- ***Cooling time scale***:
  - In the HIM with temperatures $T \sim 10^6 - 10^7\,\mathrm{K}$, the cooling time scale is:

$$\boxed{t_{\rm cool} = \frac{5}{2}\frac{nkT}{\Lambda} \qquad \begin{aligned} n_e &\approx 1.2 n_{\rm H} \\ n &\approx 2.3 n_{\rm H} \end{aligned}}$$

For fully ionized gas,
one electron from an ionized hydrogen
two electrons from a doubly-ionized helium.

  - The cooling time at $T \sim 10^6\,\mathrm{K}$ is

$$t_{\rm cool} = \frac{5}{2}\frac{2.3}{1.2}\frac{kT}{\Lambda/(n_e n_{\rm H})}\frac{1}{n_{\rm H}}$$

$$\begin{aligned} t_{\rm cool} &\approx 48\,[\mathrm{Myr}]\; T_6^{1.7} \left(\frac{n_{\rm H}}{0.004\,\mathrm{cm}^{-3}}\right)^{-1} \\ &\approx 0.19\,[\mathrm{Myr}]\; T_6^{1.7} \left(\frac{n_{\rm H}}{1\,\mathrm{cm}^{-3}}\right)^{-1} \end{aligned}$$

$$\longleftarrow \qquad \Lambda/n_e n_{\rm H} \approx 1.1 \times 10^{-22}\, T_6^{-0.7}\;[\mathrm{erg\,cm^3\,s^{-1}}]$$

The HIM frequently doesn't have time to cool thoroughly before another supernova shock wave comes through to heat it again.

  - At $T \sim 10^7\,\mathrm{K}$, the cooling time is

$$\begin{aligned} t_{\rm cool} &\approx 7.2\,[\mathrm{Gyr}]\; T_7^{1/2} \left(\frac{n_{\rm H}}{0.004\,\mathrm{cm}^{-3}}\right)^{-1} \\ &\approx 29\,[\mathrm{Myr}]\; T_7^{1/2} \left(\frac{n_{\rm H}}{1\,\mathrm{cm}^{-3}}\right)^{-1} \end{aligned}$$

$$\longleftarrow \qquad \Lambda/n_e n_{\rm H} \approx 2.3 \times 10^{-24}\, T_6^{0.5}\;[\mathrm{erg\,cm^3\,s^{-1}}]$$

Given the low density of the HIM, the cooling time of gas is comparable to the age of our galaxy (~13 Gyr, only after 0.8 Gyr after the Big Bang; Xiang & Rix, 2020, Nature, 603, 599).

- ***Recombination and Ionization Time scale***:
  - If collisional ionization could somehow be turned off, the recombination time scale is

$$t_{\rm rec} = \frac{1}{n_{\rm e}\alpha_{\rm A,H}} \approx 0.6\,[{\rm Gyr}]\left(\frac{n_{\rm H}}{0.004\,{\rm cm}^{-3}}\right)^{-1} \qquad [T \approx 10^6\,{\rm K}] \qquad \alpha_{\rm A,H} \approx 1.5\times10^{-14}\,[{\rm cm}^3\,{\rm s}^{-1}]$$

  - If collisional ionization could suddenly switch on, the collisional ionization time scale is

$$t_{\rm ci} = \frac{1}{n_{\rm e}k_{\rm ci}} \approx 160\,[{\rm yr}]\left(\frac{n_{\rm H}}{0.004\,{\rm cm}^{-3}}\right)^{-1} \qquad [T \approx 10^6\,{\rm K}] \qquad k_{\rm ci,H} \approx 5.0\times10^{-8}\,[{\rm cm}^3\,{\rm s}^{-1}]$$

  - These times scales indicates that
    - If cold neutral hydrogen gas is shock-heated to ~ 10^6 K in a time $t_{\rm heat} \ll t_{\rm ci}$, it will take a time $t \sim t_{\rm ci}$ for hydrogen to become ionized. ***During this time interval, the hydrogen will be out of collisional ionization equilibrium (under-ionized than in CIE).***
    - If highly ionized gas at ~ 10^6 K is cooled on a timescale $t_{\rm cool} \ll t_{\rm rec}$, and the heating source is turned off, it will take a time $t \sim t_{\rm rec}$ for the hydrogen to recombine. ***During the intervening time, the gas will be out of CIE (over-ionized than in CIE).*** This is sometimes called ***"delayed recombination"***.

- If a gradually cooling gas of the HIM to be remained in CIE, we require $t_{\rm rec} < t_{\rm cool}$

Assuming the recombination rate coefficient at high temperatures

$$\alpha_{\rm A,H} \approx 5 \times 10^{-16}\,{\rm cm^3\,s^{-1}}\left(T/10^7\,{\rm K}\right)^{-1.5}$$

- At T ~ 10^6 K,

$$t_{\rm rec} = \frac{1}{n_e \alpha_{\rm A,H}} \approx 0.5\,[{\rm Gyr}]\,\left(T/10^6\,{\rm K}\right)^{1.5}\left(n_{\rm H}/0.004\,{\rm cm^{-3}}\right)^{-1}$$

$$t_{\rm cool} \approx 48\,[{\rm Myr}]\,T_6^{1.7}\left(n_{\rm H}/0.004\,{\rm cm^{-3}}\right)^{-1}$$

$$\Rightarrow \quad \frac{t_{\rm rec}}{t_{\rm cool}} \approx 10\left(T/10^6\,{\rm K}\right)^{-0.2}$$

- At T ~ 10^7 K,

$$t_{\rm rec} = \frac{1}{n_e \alpha_{\rm A,H}} \approx 16\,[{\rm Gyr}]\,\left(T/10^7\,{\rm K}\right)^{1.5}\left(n_{\rm H}/0.004\,{\rm cm^{-3}}\right)^{-1}$$

$$t_{\rm cool} \approx 7.2\,[{\rm Gyr}]\,T_7^{0.5}\left(n_{\rm H}/0.004\,{\rm cm^{-3}}\right)^{-1}$$

$$\Rightarrow \quad \frac{t_{\rm rec}}{t_{\rm cool}} \approx 2.2\left(T/10^7\,{\rm K}\right)$$

$t_{\rm rec}(T \approx 10^6\,{\rm K}) \ll t_{\rm rec}(T \approx 10^7\,{\rm K}), \;\; t_{\rm cool}(T \approx 10^6\,{\rm K}) \ll t_{\rm cool}(T \approx 10^7\,{\rm K})$

Therefore, ***in the extremely hot regions, the hotter the gas is, the further away it is from CIE.***

Cooling in Shocked Gas

- ***The hot shocked gas is out of equilibrium***, and will start to cool. Thus, the shock will be followed by a radiative zone in which the shock heated gas cools down by radiating away photons.
    - At high temperatures $T > 2 \times 10^7\,\mathrm{K}$
        - The cooling is dominated by bremsstrahlung (free-free radiation), for which the *specific* cooling rate (per mass) is

$$\mathcal{L} = 2.7\,[\mathrm{erg\,g^{-1}\,s^{-1}}]\left(\frac{T}{10^7\,\mathrm{K}}\right)^{1/2}\left(\frac{n_\mathrm{H}}{1\,\mathrm{cm^{-3}}}\right)$$

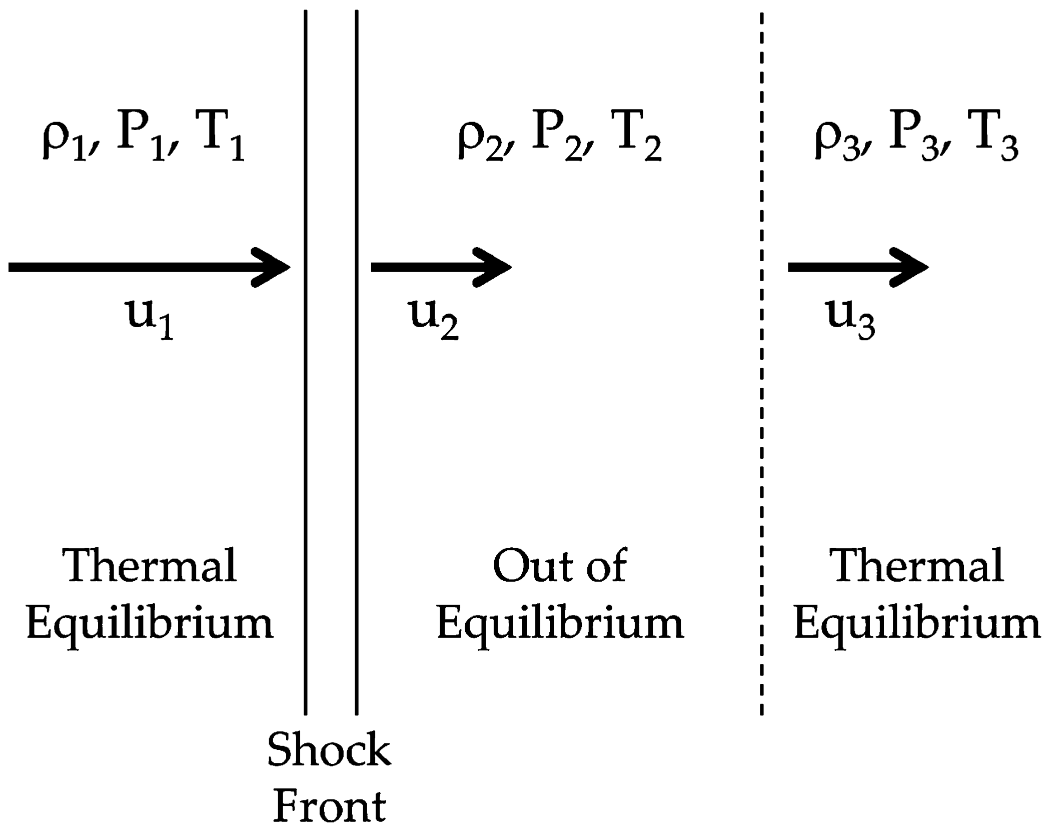
assuming a gas of fully ionized hydrogen. The *specific* internal energy of ionized hydrogen (per mass) is

$$\epsilon = \frac{3}{2}\frac{(2n_\mathrm{H})kT}{n_\mathrm{H}m_\mathrm{H}} \approx 2.5 \times 10^{15}\,[\mathrm{erg\,g^{-1}}]\left(\frac{T}{10^7\,\mathrm{K}}\right)$$

- Then, the bremsstrahlung cooling time is

$$\begin{aligned} t_\mathrm{cool} = \frac{\epsilon}{\mathcal{L}} &\approx 29\,[\mathrm{Myr}]\left(\frac{T}{10^7\,\mathrm{K}}\right)^{1/2}\left(\frac{n_\mathrm{H}}{1\,\mathrm{cm^{-3}}}\right)^{-1} \\ &\approx 34\,[\mathrm{Myr}]\left(\frac{V_\mathrm{s}}{1000\,\mathrm{km\,s^{-1}}}\right)\left(\frac{n_\mathrm{H}}{1\,\mathrm{cm^{-3}}}\right)^{-1} \end{aligned}$$

$$T \simeq \frac{3}{16}\frac{m}{k}V_s^2$$



The structure of a plane parallel radiative shock [Figure 5.3 Ryden]

- During this time, the gas will move a distance, relative to the shock front:

$$R_{\rm cool} \approx u_2 t_{\rm cool} \approx \frac{u_1}{4} t_{\rm cool}$$
$$\approx 8.7\,[{\rm kpc}] \left(\frac{V_s}{1000\,{\rm km\,s^{-1}}}\right)^2 \left(\frac{n_{\rm H}}{1\,{\rm cm^{-3}}}\right)^{-1}$$

  This implies that ***the approximate thickness of the radiative zone for a strong shock is a long distance compared to the scale height of the ISM in our galaxy***. Thus, the hot gas produced by high-speed shocks doesn't have time to cool before the shock runs out of gas to shock. ***<< No cooling of the shock gas in our galaxy>>***

- At lower temperature $(10^5\,{\rm K} < T < 2\times 10^7\,{\rm K})$, corresponding to slower shock speeds

  $(80\,{\rm km\,s^{-1}} < u_1 = V_s < 1200\,{\rm km\,s^{-1}})$

  - The collisionally excited lines do most of the cooling. A useful approximation for the cooling rate gives

$$t_{\rm cool} \approx 6600\,[{\rm yr}] \left(\frac{V_s}{100\,{\rm km\,s^{-1}}}\right)^{3.4} \left(\frac{n_{\rm H}}{1\,{\rm cm^{-3}}}\right)^{-1}$$

  - This yields a thickness for the radiative zone.

$$R_{\rm cool} \approx \frac{V_s}{4} t_{\rm cool} = 0.17\,[{\rm pc}] \left(\frac{V_s}{100\,{\rm km\,s^{-1}}}\right)^{4.4} \left(\frac{n_{\rm H}}{1\,{\rm cm^{-3}}}\right)^{-1}$$

  - These shorter time scales and length scales mean that ***radiative cooling is more effective at changing the structure of slower shocks.***

Homework (due date: 05/22)

[Q13]

The "cooling time" $\tau_{\rm cool} \equiv |d\ln T/dt|^{-1}$. Suppose the power radiated per unit volume Λ can be approximated by

$$\Lambda \approx A n_{\rm H} n_e \left[T_6^{-0.7} + 0.021 T_6^{1/2}\right]$$

for gas of cosmic abundances, where $A = 1.1 \times 10^{-22}\,{\rm erg\,cm^3\,s^{-1}}$, and $T_6 \equiv T/10^6\,{\rm K}$. Assume the gas to have $n_{\rm He} = 0.1 n_{\rm H}$, with both H and He fully ionized.

Compute the cooling time (at constant pressure) due to radiative cooling

(a) in a supernova remnant at $T = 10^7\,{\rm K}$, $n_{\rm H} = 10^{-2}\,{\rm cm^{-3}}$.

(b) for intergalactic gas within a dense galaxy cluster (the "intracluster medium") with $T = 10^8\,{\rm K}$, $n_{\rm H} = 10^{-3}\,{\rm cm^{-3}}$.

[Q14]

Consider a strong shock wave propagating into a medium that was initially at rest. Assume the gas to be monatomic ($\gamma = 5/3$). Consider the material just behind the shock front. The gas has an energy density $u_{\rm thermal}$ from random thermal motions, and an energy density $u_{\rm flow}$ from the bulk motion of the shocked gas. If cooling is negligible, calculate the ratio $u_{\rm flow}/u_{\rm thermal}$ in the frame of reference where the shock front is stationary.