복사전달 특강

Special Topics in Radiative Transfer

Lecture 6 - 2026 April 7 (Tuesday), 1PM
Lecture 7 - 2026 April 14 (Tuesday), 1PM

updated on 04/19

Based on: Chapter 2 of Vassiliev (Monte Carlo Methods for Radiation Transport, 2017)
Chapter 5 of Noebauer & Sim (Living Reviews in Computational Astrophysics)

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1. Uniform Distribution & Pseudorandom Numbers

Multiplicative Congruential Algorithm

$$x_{i+1} = (a \cdot x_i + c) \mod m$$
$$\gamma_{i+1} = \frac{x_{i+1}}{m}$$

Parameters:

a = multiplier
m = modulus
x_0 = seed

$$a = 1103515245$$
$$c = 12345$$
$$m = 2^{31}$$

All three parameters are positive integers.

Key Properties

- Generates $\gamma \in U(0,1)$
- Deterministic (reproducible with same seed)
- Finite period — modern generators have very long periods
- Seed can be set randomly (e.g. system clock)
- For further reading: Fishman (1996)

Sample from Uniform(a, b)

Problem: generate ξ ~ Uniform(a, b)

1. Generate $\gamma \in U(0,1)$
2. $\xi = (b - a)\gamma + a$

Uniform Point in 3D Rectangular Box

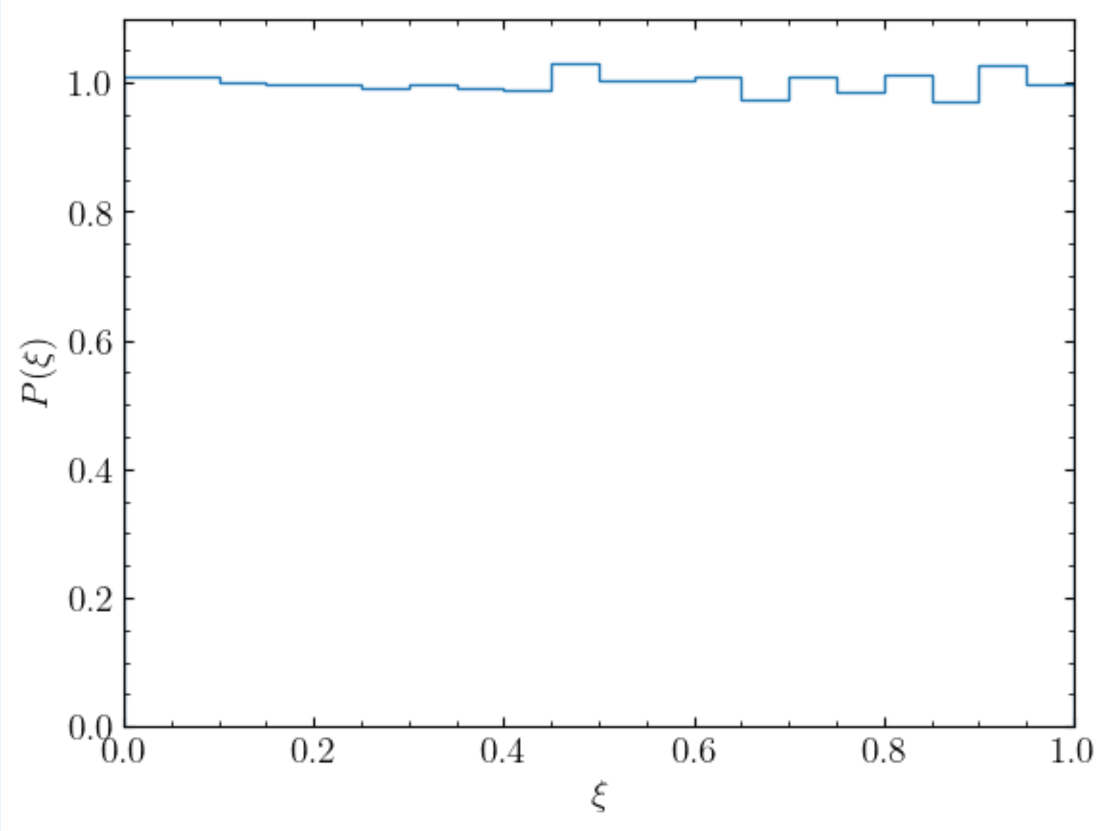
Box: $[x_a, x_b] \times [y_a, y_b] \times [z_a, z_b]$

Generate γ → $\xi = (x_b - x_a)\gamma + x_a$
Generate γ → $\eta = (y_b - y_a)\gamma + y_a$
Generate γ → $\zeta = (z_b - z_a)\gamma + z_a$

The algorithm uses three different and statistically independent γs.

Python Code

```
import numpy as np
import matplotlib.pyplot as plt

#———
def simplest_random(seed=None):
    global seed_g
    if seed == None:
        try:
            _ = seed_g
        except:
            seed_g = 110311
        a, c, m = 1103515245, 12345, 2**31
        x2 = np.mod((a*seed_g + c), m)
        seed_g = x2
    else:
        seed_g = seed
        return 0
    return x2/m

#———————
nmax = 100000
simplest_random(seed=10100)
sample = np.zeros(nmax)
for i in range(nmax):
    sample[i] = simplest_random()

#———————
plt.hist(sample, density=True, bins=20, range=(0.0, 1.0), histtype='step')
plt.xlim(0.0, 1.0)
plt.ylim(0.0, 1.1)
plt.xlabel(r'$\xi$')
plt.ylabel(r’$P(\xi)$')
```



2. The Inversion Method for Continuous Distributions

Problem: Generate a random number ξ that has a continuous distribution $f(x)$.

Groundwork:

Find CDF of $f(x)$: $F(x) = \int_{-\infty}^{x} f(x')dx'$

Steps:

1. Generate a uniform random number γ:
   $\gamma \in U(0,1)$
2. Solve $\gamma = F(\xi)$ for ξ
   → $\xi = F^{-1}(\gamma)$

Proof

By definition,

$$F(x) = P(\xi \le x) = P\{F^{-1}(\gamma) \le x\}$$

Because $F(x)$ is monotonic,

$$= P\{F[F^{-1}(\gamma)] \le F(x)\}$$
$$= P\{\gamma \le F(x)\}$$

This proves that the algorithm samples ξ correctly.

CDF Inversion

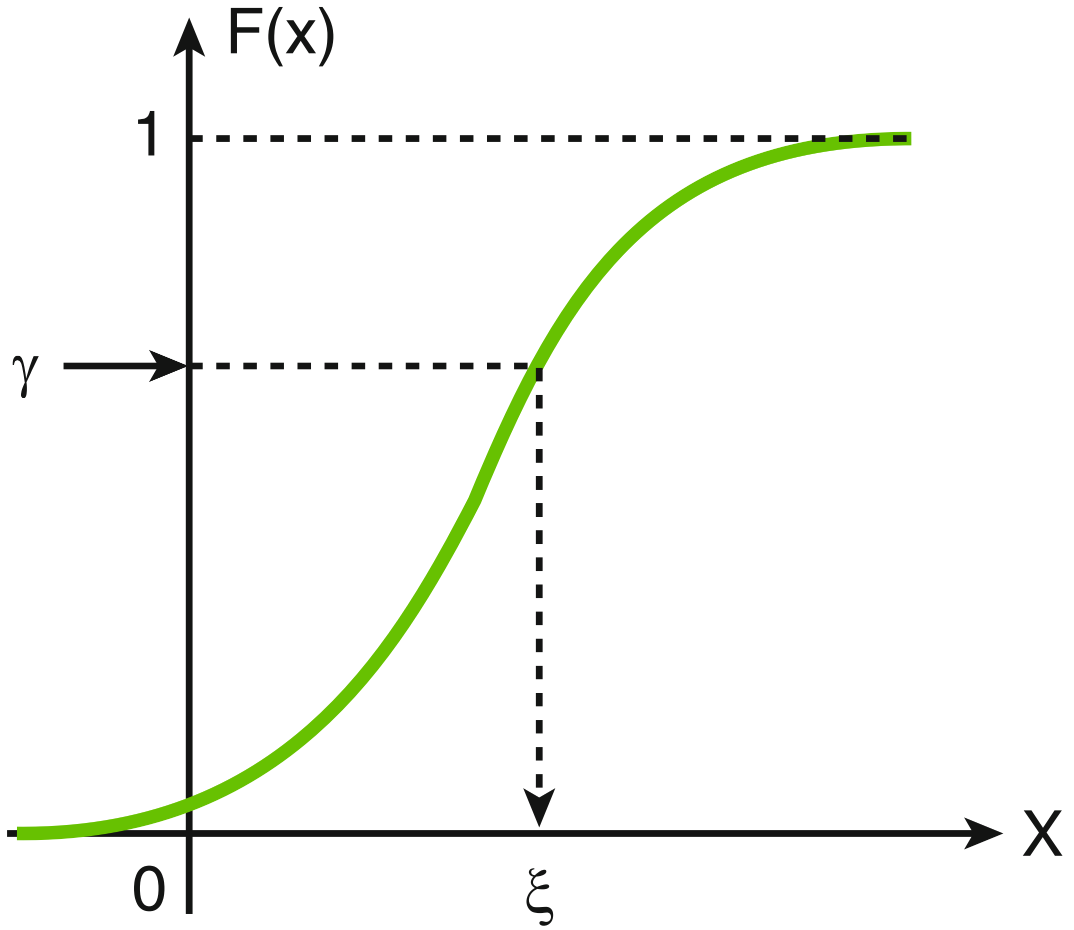
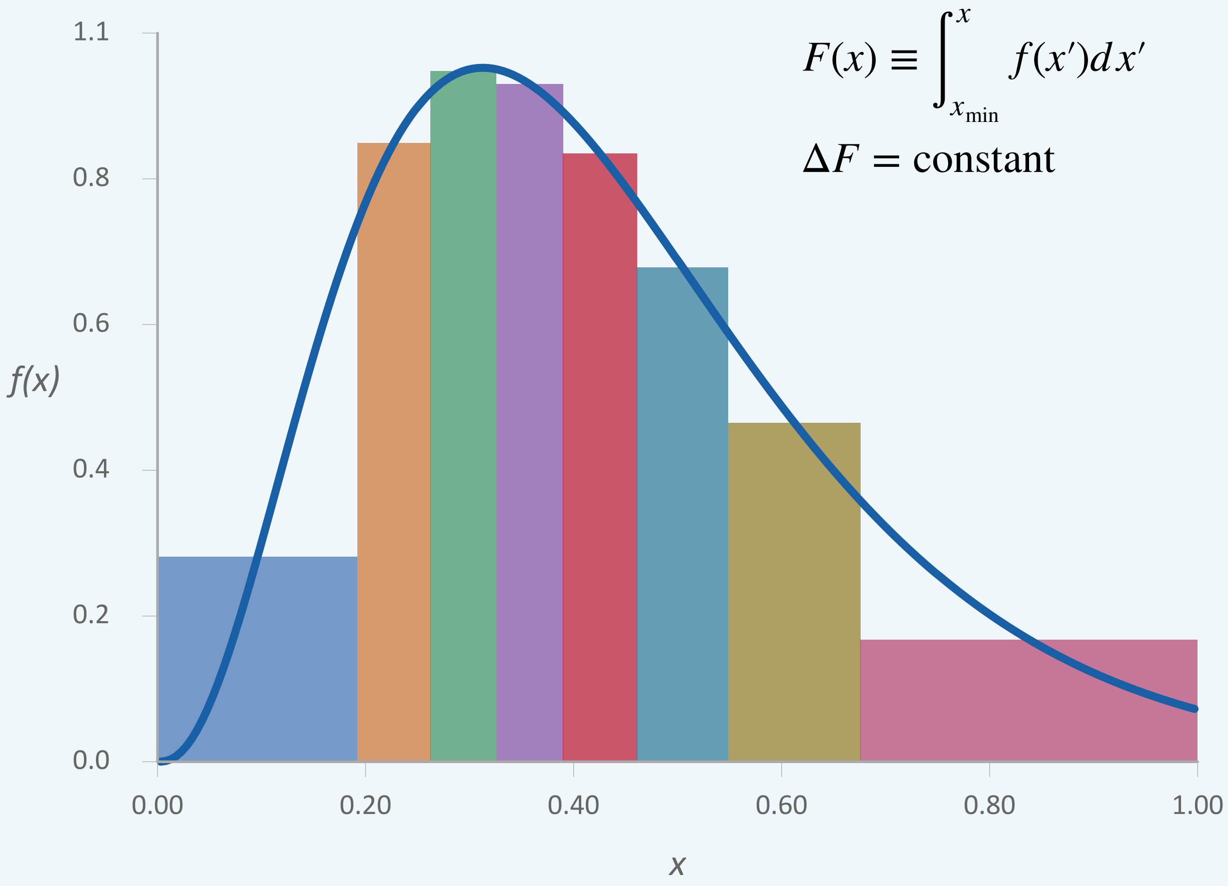
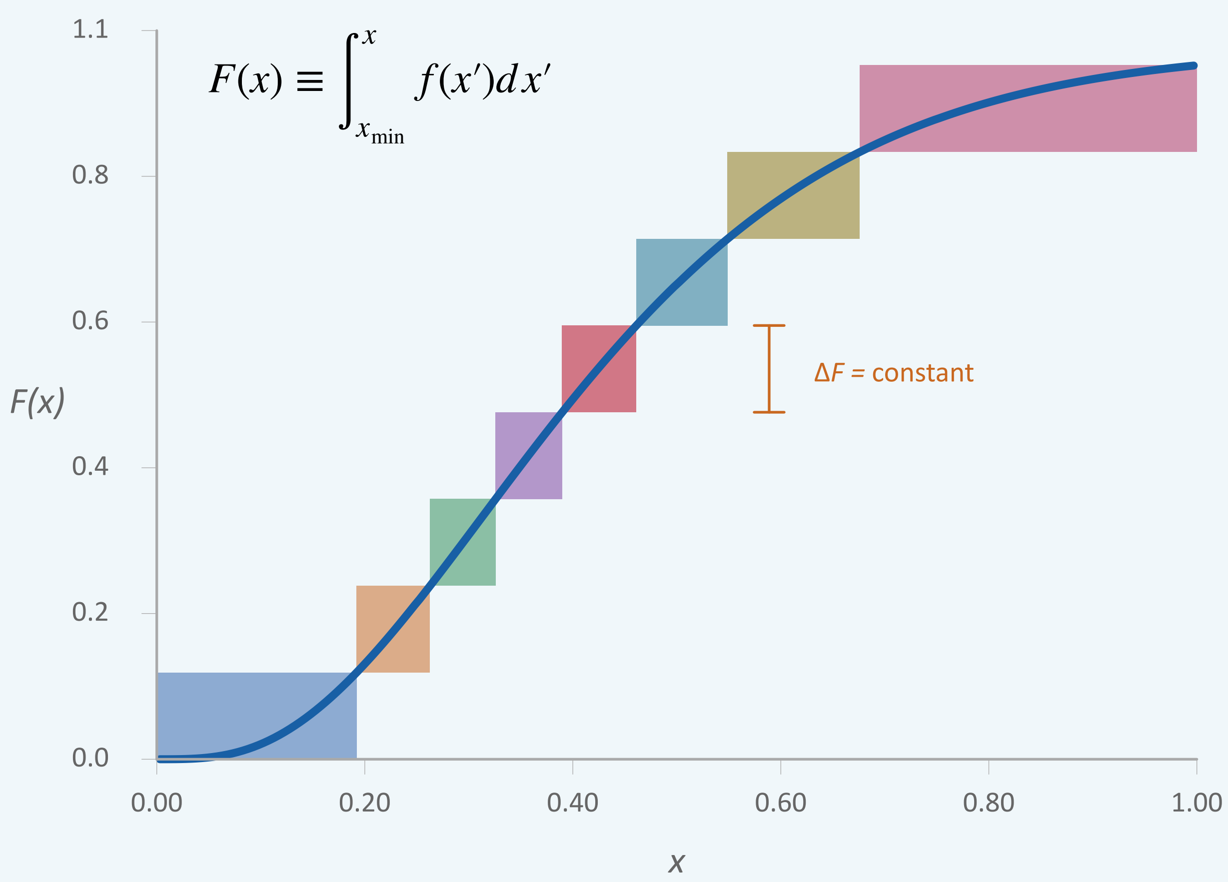


Illustration of the Inversion Method


Same color = same probability (= 1/8)


CDF with `equal' ΔF bands → maps to 'unequal' Δx

Sampling x according to PDF f(x):

- Divide the x interval $[x_{\min}, x_{\max}]$ into many small bins such that the probability of being sampled within each bins is the same; that is, $\Delta F = F(x_{i+1}) - F(x_i)$ = constant $= 1/N$ (N = number of bins).
- Then, sampling x according to the 'discrete' $f(x)$ is equivalent to sampling 'uniformly' over these unequal bins.
- In the limit of infinite number of bins, this is equivalent to sampling a uniform random γ and finding ξ such that $\gamma = F(\xi)$. In other words, $\xi = F^{-1}(\gamma)$.

Example Applications

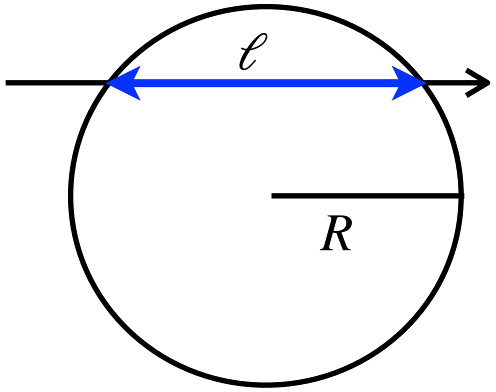
Ex 1: Chord-Length Distribution

A parallel beam of particles is incident on a sphere of unit diameter. Each particle entering the sphere travels a random path ξ (chord) within it.

Assume that the particles travel along a straight line, and do not stop within the sphere, the chord-length distribution is given by this simple formula:

$$f(x) = \begin{cases} 2x, & \text{if } 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Here, $x = \ell/(2R)$.



$$F(x) = x^2 \quad \text{for } 0 \le x \le 1$$

```
**Algorithm:**
1. Generate $\gamma \in U(0,1)$
2. $\xi = \sqrt{\gamma}$
```

Ex 2: Free Path (Exponential)

Particles travel along a straight line between points of interaction with matter. The distance between interactions is called a free path. It is distributed exponentially:

$$f(x) = \begin{cases} \sigma \exp(-\sigma x), & \text{if } x \ge 0 \\ 0, & \text{if } x < 0 \end{cases}$$

Here, σ is the optical depth per unit length (cm^{-1}) and x is the distance (cm).

$$F(x) = 1 - \exp(-\sigma x)$$

```
Algorithm:
1. Generate $\gamma \in U(0,1)$
2. $\xi = -(1/\sigma)\ln\gamma$
```

Sampling of optical depth

$$f(\tau) = \begin{cases} \sigma \exp(-\tau), & \text{if } \tau \ge 0 \\ 0, & \text{if } \tau < 0 \end{cases}$$

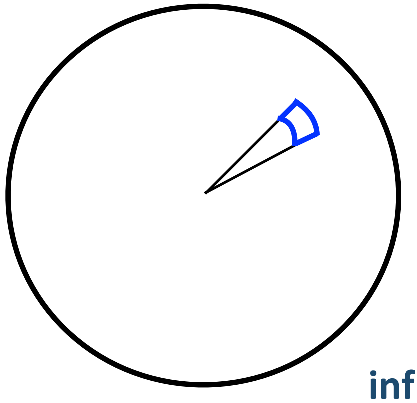
$$F(x = \tau) = 1 - \exp(-\tau)$$

```
Algorithm:
1. Generate $\gamma \in U(0,1)$
2. $\xi = -\ln\gamma$
```

Ex 1

Cross-sectional View

(The beam incident perpendicular to the slide.)



infinitesimal area = $dA = rdrd\phi$

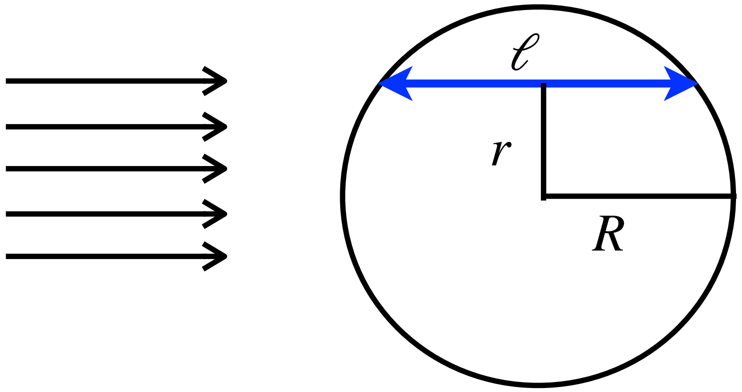
Probability for a ray to be injected within (r**,** $r + dr$**) and (**ϕ**,** $\phi + d\phi$**) is proportional to the area** $dA = rdrd\phi$**.**

$$P(r,\phi)drd\phi = \text{constant} \times dA = C \times rdrd\phi$$

$$\therefore\ P(r,\phi) = \frac{r}{\pi R^2} = \frac{1}{2\pi}\frac{2r}{R^2}$$

The probability density function for r **is then given by**

$$P(r) = \int_0^{2\pi} P(r,\phi)d\phi = \frac{2r}{R^2}$$



The chord-length is

$$\ell = 2\sqrt{R^2 - r^2}$$

$$\therefore\ \ell d\ell = -4rdr \quad \rightarrow \quad \left|\frac{dr}{d\ell}\right| = \frac{\ell}{4r}$$

The probability density function for ℓ **is then given by:**

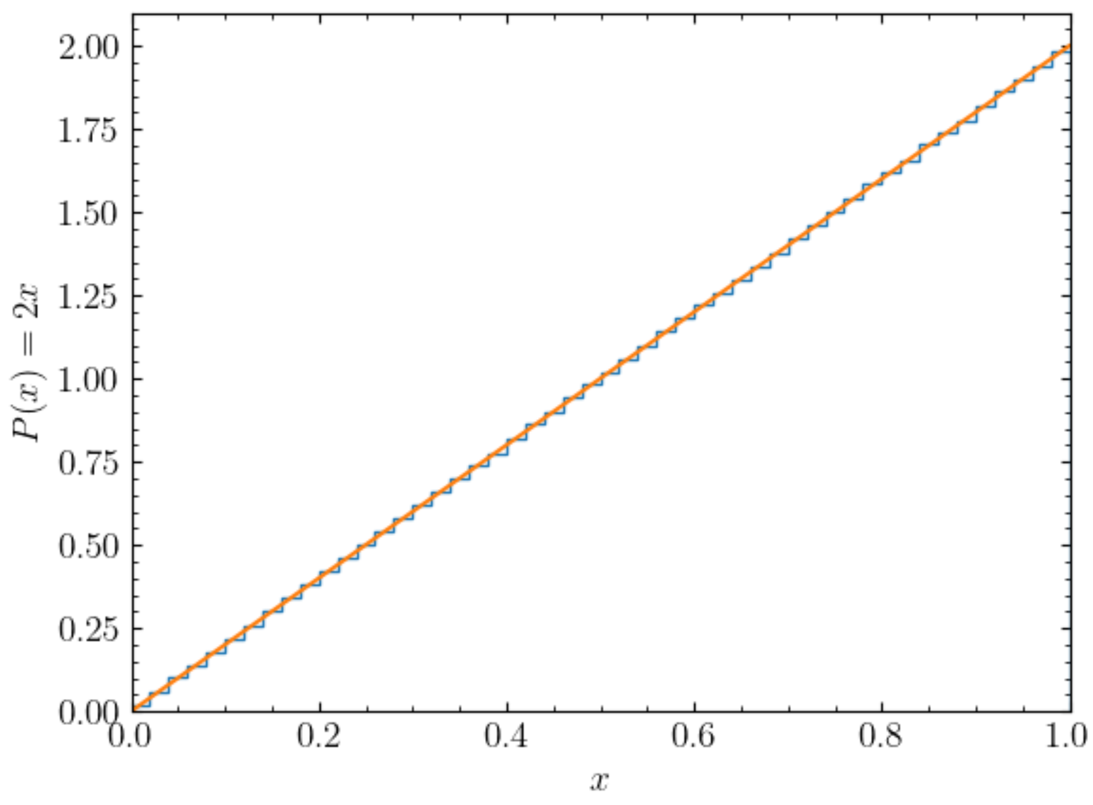
$$P(\ell) = P(r)\left|\frac{dr}{d\ell}\right| = \frac{2r}{R^2}\frac{\ell}{4r} = \frac{\ell}{2R^2}$$

The probability density function for the normalized chord length $x \equiv \frac{\ell}{2R}$ **is then:**

$$P(x) = P(\ell)\left|\frac{d\ell}{dx}\right| = 2x$$

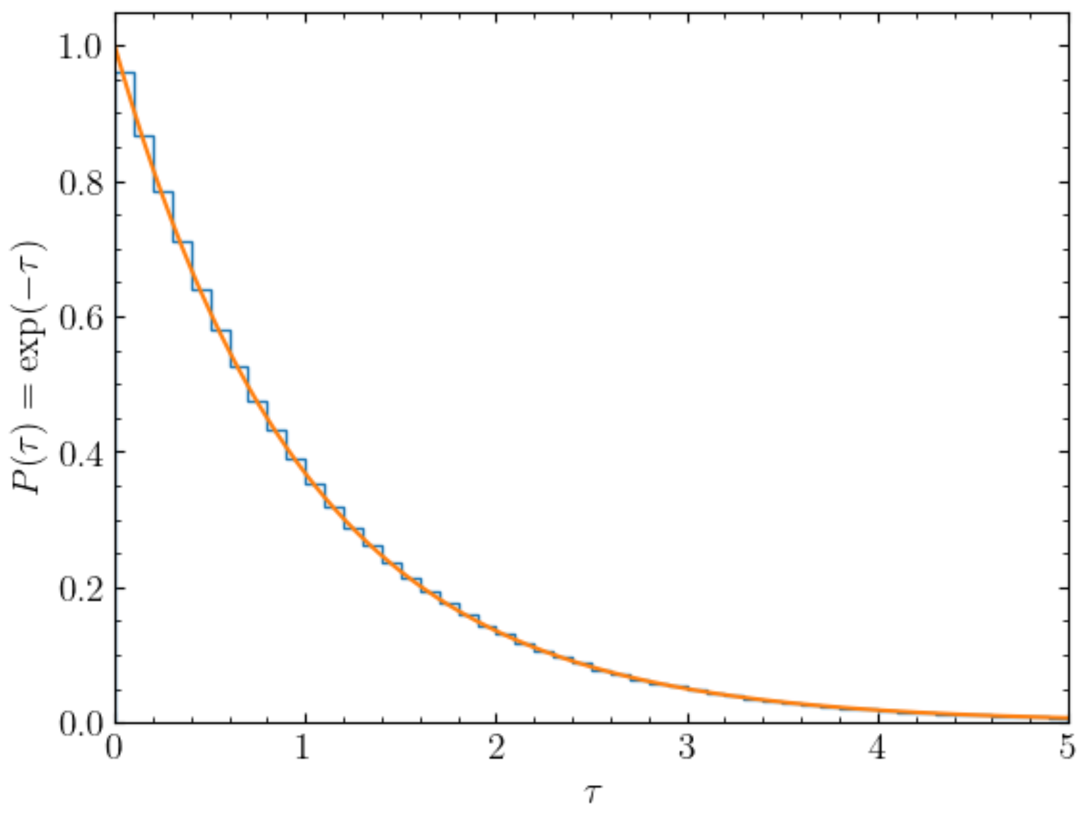
Ex 1 - Python Code

```
size = 10000000
x = np.random.random(size=size)
xi = np.sqrt(x)

plt.hist(xi, bins=50, range=(0.0,1.0), density=True, histtype='step')
xx = np.arange(0.0, 1.0+0.01, 0.01)
yy = 2.*xx
plt.plot(xx,yy)
plt.xlim(0.0,1.0)
plt.ylim(0.0, 2.1)
plt.xlabel(r'$x$')
plt.ylabel(r'$P(x)=2x$')
```



Ex 2 - Python Code

```
size = 10000000
x = np.random.random(size=size)
tau = -np.log(x)

plt.hist(tau, bins=50, range=(0.0,5.0), density=True, histtype='step')
xx = np.arange(0.0, 5.0+0.01, 0.01)
yy = np.exp(-xx)
plt.plot(xx,yy)
plt.xlabel(r'$\tau$')
plt.ylabel(r'$P(\tau)=\exp(-\tau)$')
plt.xlim(0.0, 5.0)
plt.ylim(0.0, 1.05)
```



3. The Inversion Method for Sampling Discrete Distributions

Discrete CDF Inversion

- For discrete random numbers $\{x_1, x_2, \cdots\}$, the CDF is a stepwise-constant function
- Example — die roll:
  $x_k = k$, and $f(k) = 1/6$ for $k = 1,\cdots,6$
  $F(k) = k/6$

Generic Algorithm:

```
(1) $k = 1$
    Sample $\gamma$;   $\delta = \gamma$

(2) Compute $P_k$;   $\delta = \delta - P_k$

(3) If $\delta \leq 0$ → output $\xi = x_k$, exit
    Else   $k \Rightarrow k+1$;   repeat
```

Useful Recurrence Relations:

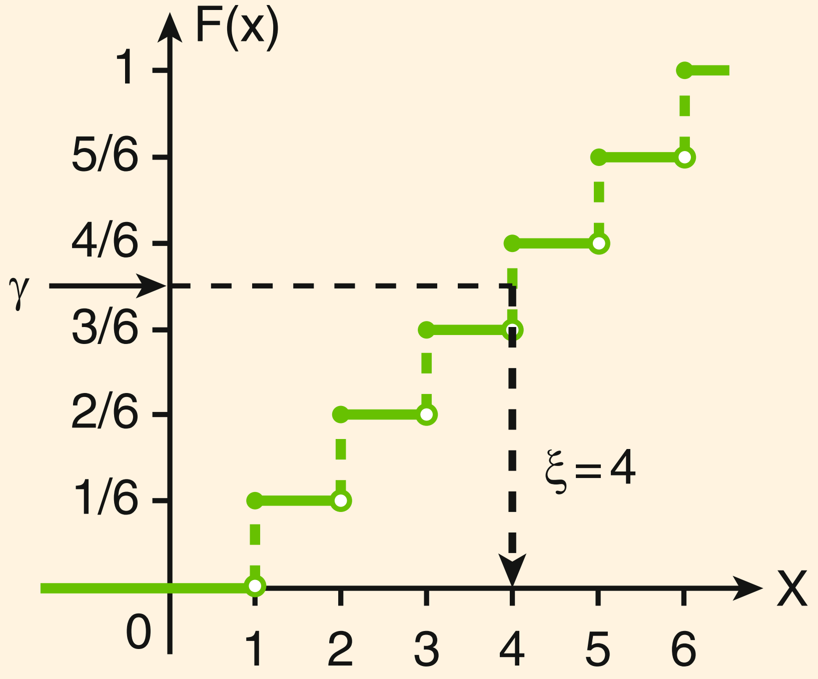
Binomial:

$$P_{k+1} = \frac{n-k}{k+1}\frac{p}{1-p}P_k; \quad P_0 = (1-p)^n; \;\; k = 0,1,\cdots$$

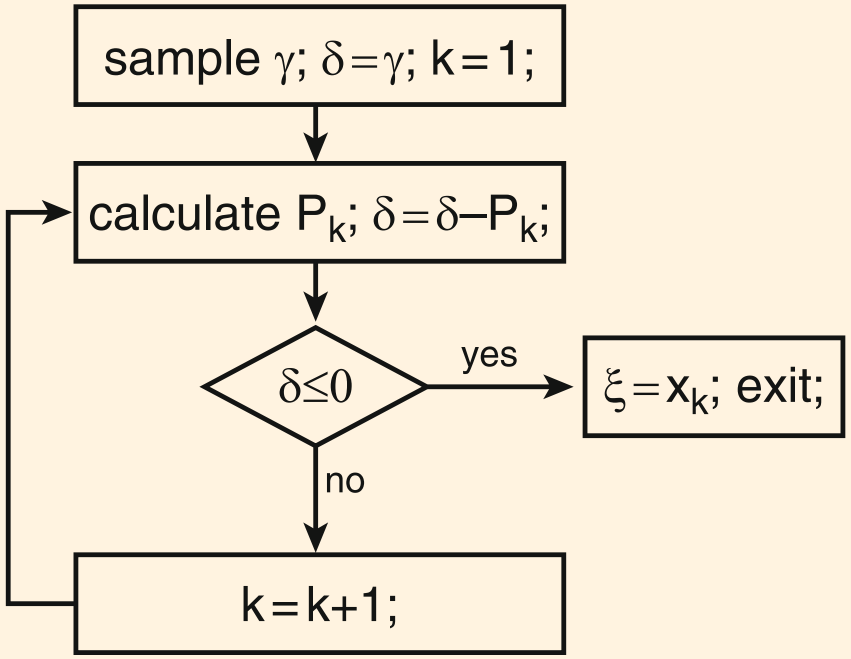
Poisson:

$$P_{k+1} = \frac{\mu}{k+1}P_k; \;\; P_0 = \exp(-\mu); \;\; k = 0,1,\cdots$$

Inversion of the CDF for one roll of a die



Algorithm



sample γ; δ=γ; k=1;

calculate P_k; δ=δ−P_k;

δ≤0

yes

ξ=x_k; exit;

no

k=k+1;

Numerical Inversion

- The inversion transformation techniques is not always feasible.
- However, even if determining the cumulative distribution function is analytically challenging, it can be done by means of numerical integration and values for $F(x)$ pre-calculated for a number of monotonically increasing x_i.

Algorithm

- Pre-calculate the cumulative distribution function $F(x_i)$ values for a number of monotonically increasing x_i.
- Once these values, $F(x_i)$, are available, the distribution can be sampled by first selecting a grid interval $[x_i, x_{i+1}]$ according to

$$i = \max\{j;\ F(x_j) \leq \gamma\}$$

- Since γ now lies between $F(x_i)$ and $F(x_{i+1})$, the final sampling is performed by linear interpolation

$$x = x_{i+1} - \frac{F(x_{i+1}) - \gamma}{F(x_{i+1}) - F(x_i)}(x_{i+1} - x_i)$$

4. Simple Rejection Method

The CDF inversion method is not always applicable to every distribution, nor is it necessarily the fastest or simplest method. The rejection method is a good alternative that offers a very straightforward algorithm.

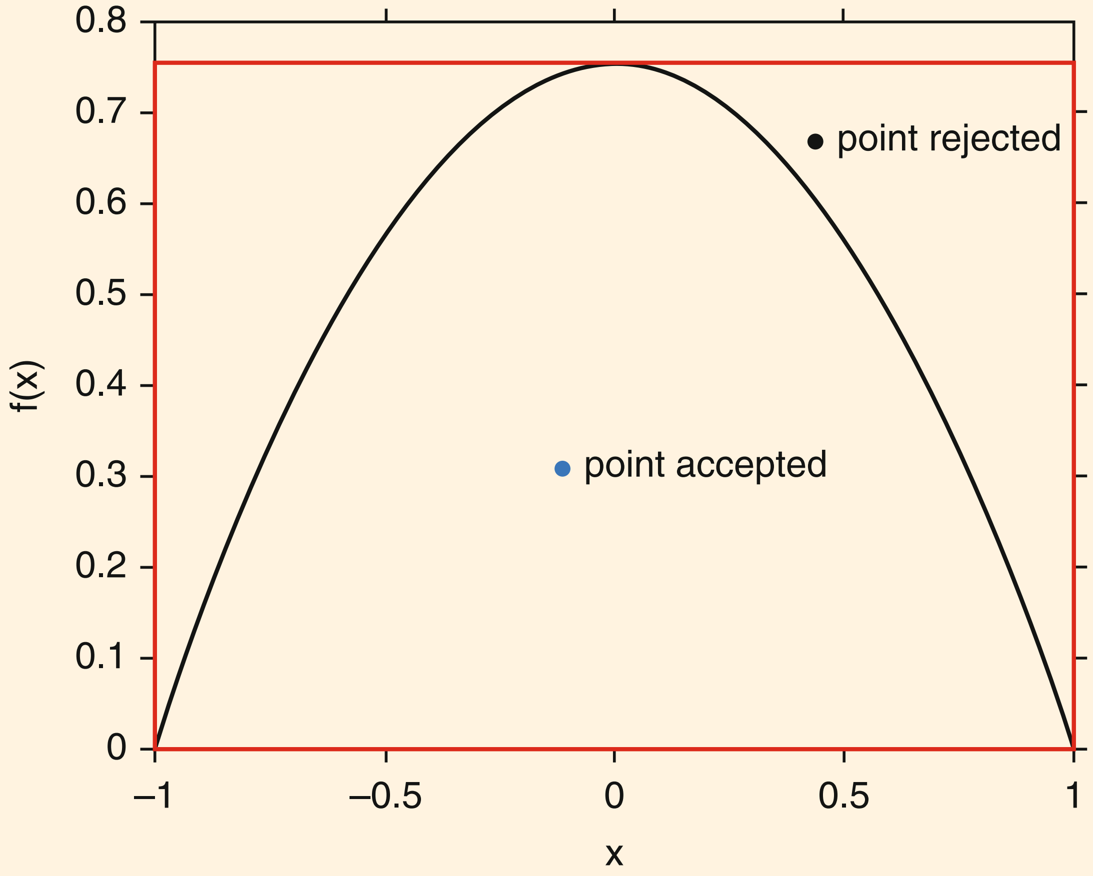
- Find minimum bounding box for f(x) (the red box)
- Place random point uniformly in box
- Accept if point falls below f(x) curve

Algorithm:

```
1. Sample $\xi$ ~ Uniform[$x_{\min}$, $x_{\max}$]
2. Sample $\eta$ ~ Uniform[0, max $f(x)$]
3. If $\eta < f(\xi)$  → ACCEPT $\xi$
   Else          → REJECT, repeat
```

The efficiency of the algorithm, defined as the probability of accepting a point, is

Efficiency = Area under f(x) / Box area



Example:

$f(x) = (3/4)(1 - x^2)$ for $-1 \leq x \leq 1$

(1) Sample ξ uniformly within [−1,1],
and η uniformly within [0, 3/4].

(2) If $\eta > (3/4)(1 - \xi^2)$, then the point is rejected,
go back to step 1.
Else, the point is accepted; output ξ.

Efficiency = 2/3 ≈ 66.7%

5. Neumann's Method (Rejection with Proposal Distribution)

Algorithm Setup

Choose proposal $g(x)$ such that:

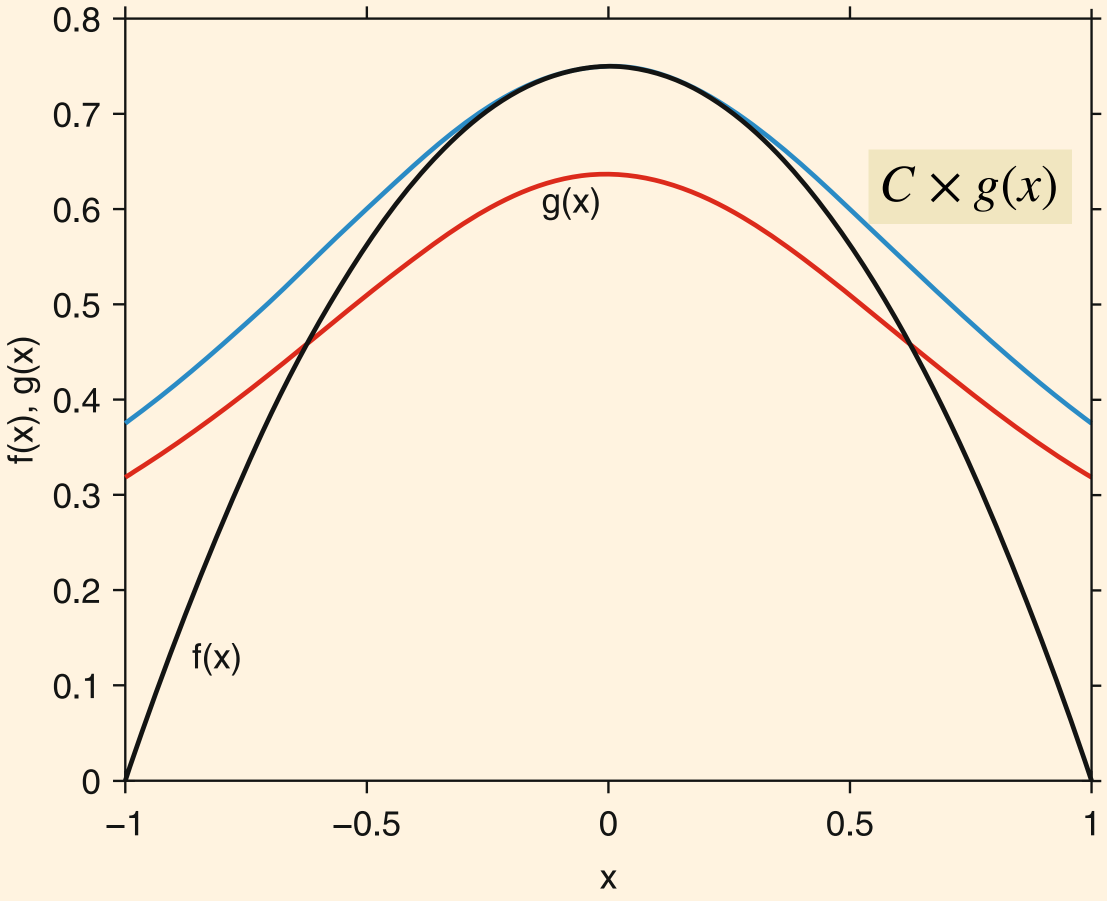
(a) $g(x)$ is similar to $f(x)$
(b) $g(x)$ is easy to sample
(c) A number $C \geq 1$ exists such that $Cg(x) \geq f(x)$ for all x
→ Choose $g(x)$ with the smallest C.

Algorithm Steps:

```
1. Sample $\xi$ from $g(x)$
2. Generate a uniformly-random number $\gamma$
3. If $\gamma \leq f(\xi)/[C \times g(\xi)]$
     → output $\xi$ and exit
   Else → go back to step 1
```

Efficiency = Area under $f(x)$ / Area under $C \cdot g(x)$
= 1/C

If g(x) = const → same as simple rejection
If g(x) = f(x) → C=1, no rejection needed



Example Application

Sample from $f(x) = (3/4)(1 - x^2)$**,** $-1 \le x \le 1$

Choose proposal distribution:

$$g(x) = \frac{2}{\pi}\frac{1}{1+x^2} \quad (|x| \le 1\text{; ‘truncated’ Cauchy distribution})$$

C = 3π/8 ≈ 1.178
Efficiency ≈ 85%
(vs 67% for simple rejection)

Sampling formula for $g(x)$**:**

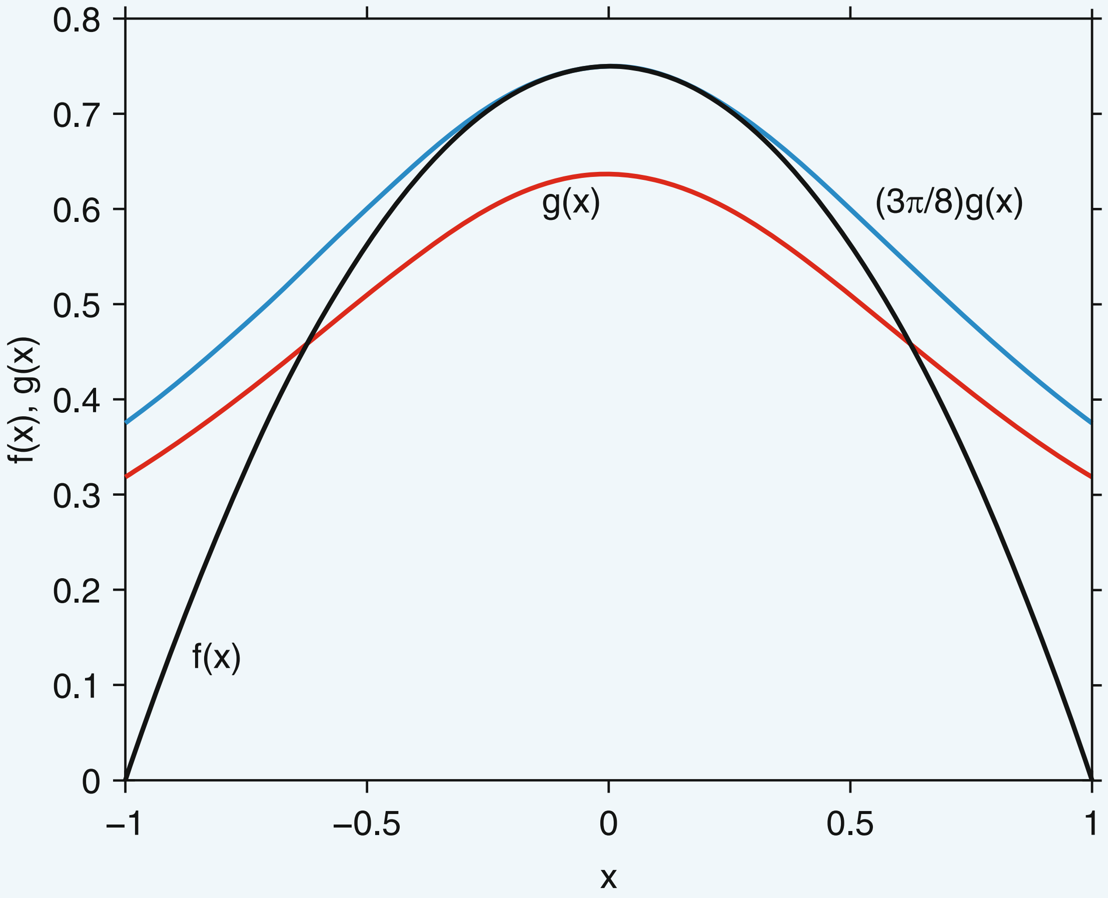
$$\xi = \tan\left[\frac{\pi}{2}\left(\gamma - \frac{1}{2}\right)\right]; \gamma \in U(0,1)$$

$$\Leftarrow \int_{-1}^{\xi} \frac{2}{\pi}\frac{dx}{1+x^2} = \frac{2}{\pi}\left(\tan^{-1}\xi + \frac{\pi}{4}\right) = \gamma$$

Acceptance test:

$$f(x)/[Cg(x)] = 1 - x^4$$

$$\therefore\ \gamma \le 1 - \xi^4$$



Algorithm for Cauchy distribution

$$P(x) = \frac{1}{\pi}\frac{1}{1+x^2}, \text{ where } -\infty \le x \le \infty$$

$$\int_{-\infty}^{\xi} \frac{1}{\pi}\frac{dx}{1+x^2} = \frac{1}{\pi}\left(\tan^{-1}\xi + \frac{\pi}{2}\right) = \gamma \quad \text{for } \gamma \in U(0,1)$$

Therefore, $\xi = \tan\left[\pi\left(\gamma - \frac{1}{2}\right)\right]$

6. Variable Transformation

It may be possible to transform a random variable ξ into another variable $\eta = g(\xi)$ that has a distribution that is easier to sample.
Then, first sample η and then perform the inverse transformation $\xi = g^{-1}(\eta)$.

$$g_\eta(y) = f_\xi(x)\left|\frac{dx}{dy}\right| = \frac{f_\xi(x)}{\left|dg(x)/dx\right|}$$

The right-hand side should be expressed as a function of y using $x = g^{-1}(y)$.

This method is also applicable to multidimensional variables.

$$g_{\eta_1,\cdots\eta_n}(y_1,\cdots,y_n) = f_{\xi_1,\cdots,\xi_n}(x_1,\cdots,x_n)\left|\det\left\{\frac{\partial x_i}{\partial y_j}\right\}\right|, \quad i,j = 1,2,\cdots,n$$

Example:

Sample distribution

$$f_\xi(x) = 2xe^{-x^2}, \quad x \geq 0$$

Instead, we sample $\eta = \xi^2$ (i.e., $y = x^2$).
The distribution of η is

$$\rightarrow g_\eta(y) = f_\xi(x)\frac{1}{2x} = e^{-y}$$ (exponential!)

Algorithm:

1. Sample η from $\exp(-y)$ for $y \geq 0$ via the CDF inversion method:
   $\eta = -\ln(\gamma). \;\Leftarrow \gamma \in U(0,1)$

2. Output $\xi = \sqrt{\eta} = \sqrt{-\ln(\gamma)}$

7. Sum of Distributions (Mixture Sampling)

Sample distribution

$$f(x) = \sum_i p_i f_i(x)$$

where $p_i \geq 0$, $\sum_i p_i = 1$, and $\int f_i(x)dx = 1$

Algorithm:

1. Sample a channel number i from the discrete distribution given by probabilities $\{p_1, p_2, \cdots\}$

2. Sample ξ from $f_i(x)$

Example 1 — Photon Interactions:

Sample states of all particles after the interaction of a photon with an atom. We have three channels:

1. p_1 → Photoelectric absorption
    - the photon is absorbed
    - a photoelectron is ejected
    - the atom is ionized
2. p_2 → Compton scattering
    - the photon momentum changes (deflected)
    - a Compton electron is ejected
    - the atom is ionized
3. p_3 → Pair production
    - the photon disappears
    - an electron and a positron are produced.

First, we sample interaction type (the channel number, 1, 2, or 3).

Once the type of interaction is determined, we sample the parameters of particles after the interaction using the distribution function for the selected interaction type.

Example 2 — Rayleigh Scattering Phase Function

$$P(\mu) = \frac{3}{8}\left(1 + \mu^2\right), \text{ where } \mu \equiv \cos\theta$$

The PDF can be written in the following form, to apply the algorithm for a sum of distributions:

$$\begin{aligned} P(\mu) &= \frac{3}{4}\frac{1}{2} + \frac{1}{4}\frac{3}{2}\mu^2 \\ &= \frac{3}{4}P_1(\mu) + \frac{1}{4}P_2(\mu) \end{aligned}$$

Here, note that

$$\frac{3}{4} + \frac{1}{4} = 1, \quad \int_{-1}^{1} \frac{1}{2} d\mu = 1, \quad \text{and} \quad \int_{-1}^{1} \frac{3}{2}\mu^2 d\mu = 1$$

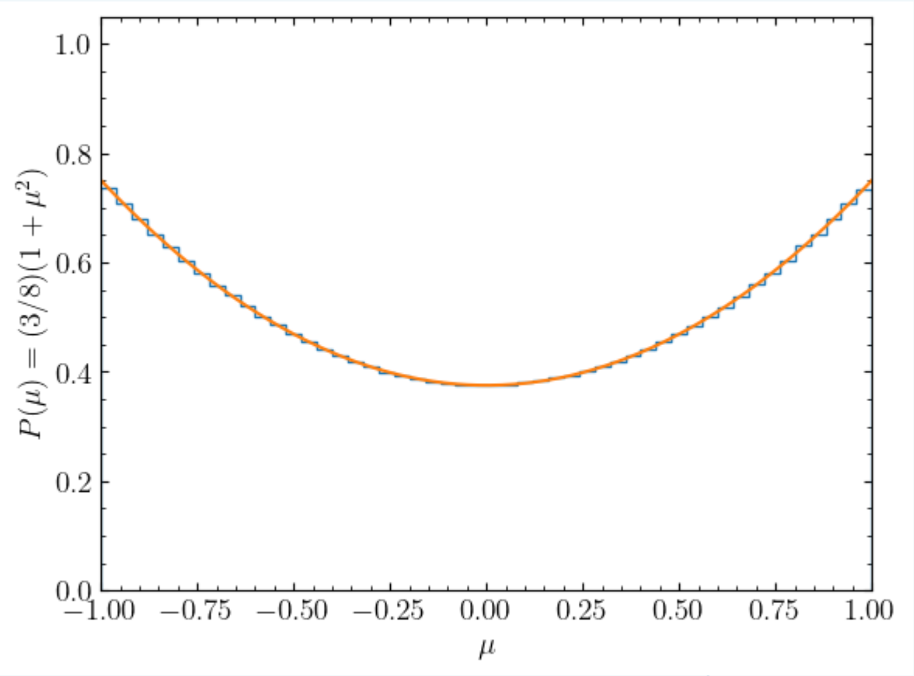
For $P_1(\mu)$, the inversion method gives:

$$\mu = 2\gamma - 1 \quad \Leftarrow \quad \gamma \in U(0,1)$$

For $P_2(\mu)$, the inversion method gives:

$$\gamma = \int_{-1}^{\mu} \frac{3}{2}x^2 dx = \frac{1}{2}\left(\mu^2 + 1\right)$$

$$\therefore\ \mu = \text{sign}(2\gamma - 1)\left|2\gamma - 1\right|^{1/3}$$



Algorithm

```
1. Generate $\gamma_1$
2. If $\gamma_1 < 3/4$, then
        Generate $\gamma_2$
        $\mu = 2\gamma_2 - 1$
   Else
        Generate $\gamma_2$
        $\mu = \text{sign}(2\gamma_2 - 1)\left|2\gamma_2 - 1\right|^{1/3}$
```

Python Code

```
size = 100000
xi = np.random.random(size=size)
w1, = np.where(xi < 3./4.)
w2, = np.where(xi >= 3./4.)

mu = np.zeros(xi.size)
q =  2.0*np.random.random(size=size) - 1.0
mu[w1] = q[w1]
mu[w2] = (np.abs(q[w2]))**(1./3.) * np.sign(q[w2])

plt.hist(mu, bins=50, range=(-1.0,1.0), density=True, histtype='step')
xx = np.arange(-1.0, 1.0+0.01, 0.01)
yy = 3./8.*(1.0+xx**2)
plt.plot(xx,yy)
plt.xlabel(r'$\mu$')
plt.ylabel(r'$P(\mu)=(3/8)(1+\mu^2)$')
plt.xlim(-1.0, 1.0)
plt.ylim(0.0, 1.05)
```

8. Compton Scattering — Combined Algorithm

Klein–Nishina formula:

Sample the scattering angle and energy of a photon after Compton scattering. We consider a simple case where the photon with an energy E interacts with a free electron 'at rest.' The probability distribution of photon energy (E') after scattering is given by

$$f(E') = A\left(\frac{E'}{E} + \frac{E}{E'} - \sin^2\theta\right) \text{ where } \cos\theta = 1 - \frac{1}{E'} + \frac{1}{E} \Leftarrow E' = \frac{E}{1 + E(1-\cos\theta)},$$

energy-momentum conservation

where the energies are in the units of electron rest energy, $E = \text{energy}/m_e c^2$.

The minimum and maximum E' can be found by setting $\theta = \pi$ and $\theta = 0$, respectively.

$$E'_{\text{min}} = E/(1+2E);\ \ E'_{\text{max}} = E$$

Strategy (3 Methods Combined; Neumann's method + a sum of two distributions + CDF inversion)

- Decompose $f(E')$ into $Cg(E')h(E')$:

$$f(E') = A\left(\frac{E'}{E} + \frac{E}{E'}\right)\left(1 - \frac{\sin^2\theta}{E'/E + E/E'}\right)$$

$$g(E') \propto \frac{E'}{E} + \frac{E}{E'} \quad \text{and} \quad h(E') = \left(1 - \frac{\sin^2\theta}{E'/E + E/E'}\right) \le 1$$

$$\Rightarrow f(E') \le A\left(\frac{E'}{E} + \frac{E}{E'}\right) = Cg(E')$$

- Sampling of the proposal distribution $g(E')$: $\quad Cg(E') = A\left(\frac{E'}{E} + \frac{E}{E'}\right)$

  This function can be written as

  $$Cg(E') = A(\alpha_1 + \alpha_2)\left[\frac{\alpha_1}{\alpha_1 + \alpha_2} f_1(E') + \frac{\alpha_2}{\alpha_1 + \alpha_2} f_2(E')\right],$$

  where

  $f_1(E') = A_1 E'$ and $f_2(E') = \frac{A_2}{E'}$, and

  their normalization constants A_1 and A_2 are obtained from $A_1 \int_{E'_{\min}}^{E'_{\max}} E' dE' = 1$ and $A_2 \int_{E'_{\min}}^{E'_{\max}} \frac{dE'}{E'} = 1.$

  $$A_1 = \frac{2}{E'^{\,2}_{\max} - E'^{\,2}_{\min}} \qquad A_2 = \frac{1}{\ln(E'_{\max}/E'_{\min})}$$

  Then, it can be determined that

  $\alpha_1 = \frac{1}{A_1 E}$ and $\alpha_2 = \frac{E}{A_2}$

  Now, we choose

  $g(E') = \frac{\alpha_1}{\alpha_1 + \alpha_2} f_1(E') + \frac{\alpha_2}{\alpha_1 + \alpha_2} f_2(E'),$ and $C = A(\alpha_1 + \alpha_2).$

- Sampling of $f_1(E')$: apply the inversion method,

  $\xi = \sqrt{(E'_{\min})^2 + 2\gamma/A_1}$ using a uniform random number $\gamma \in U(0,1)$

  $$\int_{E'_{\min}}^{\xi} A_1 E' dE' = \gamma$$

- Sampling of $f_2(E')$: apply the inversion method,

  $\xi = E'_{\min} \exp(\gamma/A_2)$ using a uniform random number $\gamma \in U(0,1)$

  $$\int_{E'_{\min}}^{\xi} \frac{E'}{A_2} dE' = \gamma$$

Full Algorithm Steps

```
1. Generate $\gamma \in U(0,1)$

2. If $\gamma < \alpha_1/(\alpha_1 + \alpha_2)$ then sample $\xi$ from $f_1(E')$:

     Generate (a new) $\gamma$
     $\xi = \sqrt{(E'_{\rm min})^2 + 2\gamma/A_1}$

   Else, sample $\xi$ from $f_2(E')$

     Generate (a new) $\gamma$
     $\xi = E'_{\rm min}\exp(\gamma/A_2)$

3. Calculate $\sin^2\theta = 1 - \left(1 - \frac{1}{\xi} + \frac{1}{E}\right)^2$ from
   Compton energy and momentum conservation
   formula

4. Generate (a new) $\gamma$

5. If $\gamma < 1 - \sin^2\theta/\left(\xi/E + E/\xi\right)$
     → output $E' = \xi$  and $\theta$;  exit

   Else, go back to step 1
```

For the Monte Carlo techniques for handling fully ‘special’ relativistic Compton scattering, see Pozdnyakov, Sobol, & Syunyaev (1983; Soviet Scientific Reviews):

https://ui.adsabs.harvard.edu/abs/1983ASPRv...2..189P/abstract

9. Superposition Method

Sample a probability distribution defined as an integral

$$f(x) = \int h(x,t)dt$$

Choose a normalized and simple-to-sample distribution $g(t)$, and rewrite the integral as

$$\int h(x,t)dt = \int g(t)\frac{h(x,t)}{g(t)}dt \quad \Rightarrow \quad \int g(t)f(x\,|\,t)dt$$

< Note that $P(A,B) = P(A\,|\,B)P(B)$>

Here, we note that this equation become equivalent to a sum of distributions when t is discretized: $\sum_i g_i f_i(x)$.

The ratio h/g can be interpreted as a conditional probability distribution. Hence, the algorithm is:

Algorithm:

1. Sample τ from distribution $g(t)$
2. Sample ξ from distribution $f(x\,|\,\tau)$

Example

Sample the following distribution:

$$f(x) = A\int_0^\infty t\exp[-t(x+t)]dt \quad (x \geq 0 \text{ and } A > 0)$$

Normalization factor:

$$\int_0^\infty f(x)dx = 1 \quad \rightarrow \quad A = \frac{2}{\sqrt{\pi}}$$

The integrand can be written as a product of two normalized distributions:

$$At\exp[-t(x+t)] = t\exp(-tx)\cdot\frac{2}{\sqrt{\pi}}\exp(-t^2)$$

Algorithm:

1. Sample τ from $(2/\sqrt{\pi})\exp(-t^2),\ t \geq 0.$
   This is the positive half of a normal distribution.
2. Sample ξ from $\tau\exp(-\tau x),\ x \geq 0.$
   This is the exponential distribution.

10. Sampling the Normal Distribution N(μ, σ²)

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

`Approach: sample δ ~ N(0,1), then output` $\xi = \sigma\delta + \mu$

Method 1: Central Limit Theorem Approximation

1. Generate $\gamma_1, \gamma_2, \cdots, \gamma_n \in U(0,1)$
2. Sum the n uniform random numbers:

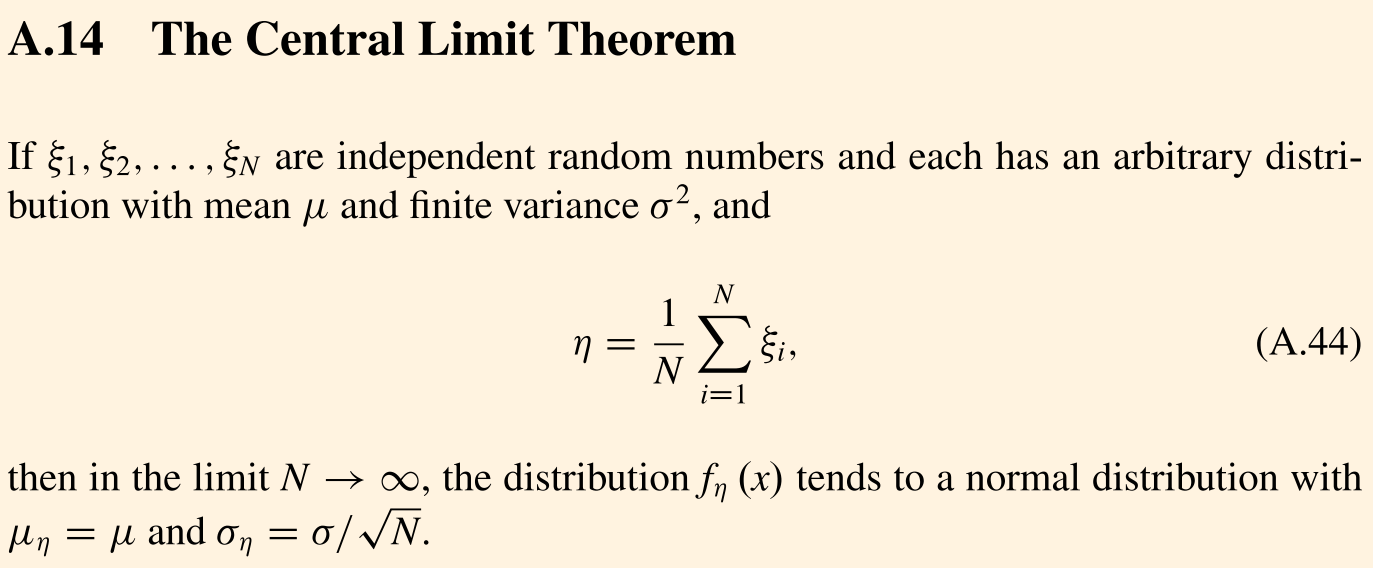
$$\xi = \sqrt{\frac{12}{n}} \sum_{i=1}^{n} \left(\gamma_i - \frac{1}{2}\right)$$

$n = 12$ has been recommended.

$$\xi = \left(\sum_{i=1}^{12} \gamma_i\right) - 6$$

It is easy to show that Var(ξ) = 1 and E(ξ) = 0.

⚠ Inaccurate in distribution tails



A.14 The Central Limit Theorem

If $\xi_1, \xi_2, \ldots, \xi_N$ are independent random numbers and each has an arbitrary distribution with mean μ and finite variance σ^2, and

$$\eta = \frac{1}{N}\sum_{i=1}^{N} \xi_i, \qquad (A.44)$$

then in the limit $N \to \infty$, the distribution $f_\eta(x)$ tends to a normal distribution with $\mu_\eta = \mu$ and $\sigma_\eta = \sigma/\sqrt{N}$.

For $n = 1$, let $\xi = \gamma - 1/2$:

$$f_\xi(x) = 1$$

$$\langle \xi \rangle = 0$$

$$\langle \xi^2 \rangle = \int_0^1 (x - 1/2)^2\, dx = \frac{1}{12}$$

$$\therefore \sigma^2 = \langle \xi^2 \rangle - \langle \xi \rangle^2 = \frac{1}{12}$$

Method 2: Box-Muller Transform

Produces 2 independent N(0,1):

1. Generate γ_1 and γ_2

2. Calculate ϕ and ρ:

$$\phi = 2\pi\gamma_1$$
$$\rho = \sqrt{-2\ln\gamma_2}$$

3. Output two samples:

$$\xi_1 = \rho\cos\phi$$
$$\xi_2 = \rho\sin\phi$$

Exact – but requires sin/cos.

This method is based on the transformation of random variables in two dimensions. Instead of one random number, we sample two independent variables, ξ_1 and ξ_2, each having distribution N(0,1). The joint probability distribution of the two variables is:

$$f(x,y) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)$$

Next, we introduce new variables, ρ and ϕ, defined as

$$x = \rho\cos\phi;\ \ y = \rho\sin\phi$$

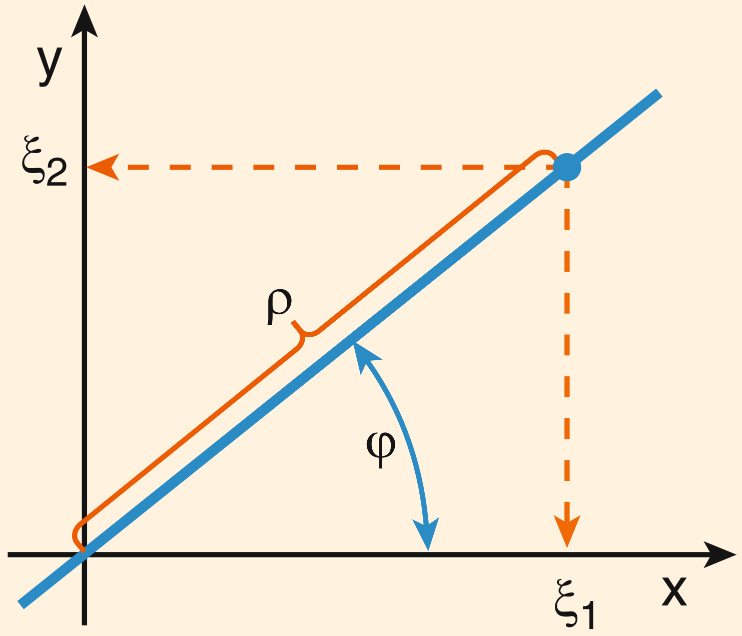
and transform the joint distribution accordingly

$$f(x,y) = \frac{1}{2\pi}\exp\left(-\frac{\rho^2}{2}\right)\rho d\rho d\phi = \frac{d\phi}{2\pi}\cdot\exp\left(-\frac{\rho^2}{2}\right)d\left(\frac{\rho^2}{2}\right)$$

This result means that the joint distribution of ϕ and $\rho^2/2$ is a product of a uniform distribution and an exponential distribution:

$$f\left(\frac{\rho^2}{2},\phi\right) = \frac{1}{2\pi}\exp\left(-\frac{\rho^2}{2}\right).$$

$$\Rightarrow\ \frac{\rho^2}{2} = -\ln\gamma$$



Method 3: Geometric Rejection

Avoid sin/cos computation:

1. Generate γ_1 and γ_2 $\in U(0,1)$
2. Sample (η_1, η_2) in unit circle:

$$\eta_1 = 2\gamma_1 - 1 \text{ and } \eta_2 = 2\gamma_2 - 1$$

3. If $\alpha^2 = \eta_1^2 + \eta_2^2 > 1$, reject
4. Else the point is within the circle.

5. Generate γ_3
6. Calculate $\rho = \sqrt{-2\ln\gamma_3}$.

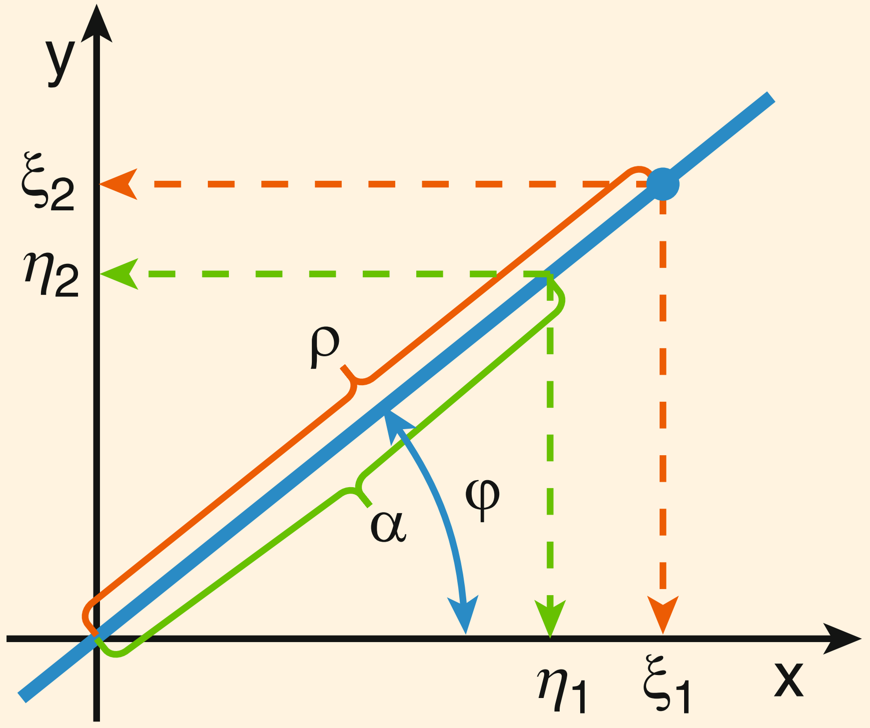
   (We sample $\rho^2/2$ from the exponential distribution).

7. Find x and y positions:

$$\xi_1 = \eta_1\rho/\alpha \text{ and } \xi_2 = \eta_2\rho/\alpha$$

Note that $\eta_1/\alpha \rightarrow \cos\phi$ and $\eta_1/\alpha \rightarrow \sin\phi$.

Efficiency = π/4 ≈ 78.5%



11. Random Points & Directions

Uniform Point within a Circle (R)

[A] Rejection:

1. Generate γ_1 and γ_2
2. $\xi_1 = R(2\gamma_1 - 1)$, $\xi_2 = R(2\gamma_2 - 1)$, where R is the radius of the circle.
3. If $\xi_1^2 + \xi_2^2 > R^2$, the the point is outside the circle; go to step 1.
4. Else, output coordinates (ξ_1, ξ_2) of the point

[B] CDF Inversion (in polar coordinates):

Polar coordinates distribution:

$$f(\rho, R) = \frac{\rho}{\pi R^2}, \quad \rho < R$$

1. Generate γ_1 → $\phi = 2\pi\gamma_1$
2. Generate γ_2 → $\rho = R\sqrt{\gamma_2}$
3. Calculate x and y coordinates of the point

$$\xi_1 = \rho\cos\phi$$
$$\xi_2 = \rho\sin\phi$$

Uniform Point in Sphere (R)

In Cartesian coordinates:

$$f(x, y, z) = \frac{1}{(4\pi/3)R^3}$$
$$\text{if } x^2 + y^2 + z^2 < R^2$$

In spherical coordinates:

$$f(r, \theta, \phi) = \frac{3r^2}{R^3}\frac{\sin\theta}{2}\frac{1}{2\pi}$$
$$\text{if } r < R$$

r, θ, and ϕ are independent:

$$f(r) = (3r^2)/R^3$$
$$f(\theta) = \left(\sin\theta\right)/2 \quad \text{or } f(\mu) = \frac{1}{2} \text{ for } \mu = \cos\theta$$
$$f(\phi) = 1/(2\pi)$$

1. Generate γ → $r = R \cdot \gamma^{1/3}$
2. Generate γ → $\mu = 2\gamma - 1$
3. Generate γ → $\phi = 2\pi\gamma$
4. $(\xi_1, \xi_2, \xi_3) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$

Isotropic Direction $\hat{\Omega}$

Consider a unit direction vector $\overrightarrow{\Omega} = (\Omega_x, \Omega_y, \Omega_z)$.

Isotropic means that, for any solid angle $d\omega$,

$$P(\overrightarrow{\Omega} \in d\omega) = \frac{d\omega}{4\pi}$$

Since an infinitesimal solid angle is

$$d\omega = \sin\theta d\theta d\phi = d\mu d\phi, \qquad (\mu \equiv \cos\theta)$$

μ and ϕ are independent and uniformly distributed:

$\mu \in [-1,1]$ uniform
$\phi \in [0,2\pi]$ uniform

Algorithm:

1. Generate γ → $\mu = 2\gamma-1$
2. Generate γ → $\phi = 2\pi\gamma$

Then, the components are:

$$\Omega_x = \left(1-\mu^2\right)^{1/2}\cos\phi$$
$$\Omega_y = \left(1-\mu^2\right)^{1/2}\sin\phi$$
$$\Omega_z = \mu$$

Uniform Random Points on a Spherical Surface

Sample an isotropic vector $\overrightarrow{\Omega} = (\Omega_x, \Omega_y, \Omega_z)$.

Calculate x, y, and z coordinates of the point:

$$\xi_1 = \Omega_x R$$
$$\xi_2 = \Omega_y R$$
$$\xi_3 = \Omega_z R$$

Here, R is the radius of the sphere.

12. Joint Distributions

Sample two random numbers that are not statistically independent.

Joint PDF factorization:

$$f_{X,Y}(x,y) = f_X(x) f_Y(y \,|\, X = x)$$

`Marginal:` $f_X(x) = \int f_{X,Y}(x,y)\,dy$

`Conditional:` $f_Y(y \,|\, X = x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)}$

Algorithm:

1. `Sample` X `from marginal` $f_X(x)$
2. `Sample` Y `from conditional` $f_Y(y \,|\, X = x)$

Example — ionization:

Sample the energy of the ejected electron first,
then sample ejection angle
(using energy-dependent conditional distribution).

13. Probability density function of a sum of independent random variables

If two random variables X and Y are independent, then the probability density of their sum is equal to the convolution of the probability densities of X and Y.

$$P_{X+Y} = \int dx P_X(x) P_Y(z-x)$$

Proof:
Independence of the two random variables implies that

$$P_{X,Y}(x,y) = P_X(x) P_Y(y).$$

The PDF for $X+Y$ is given by

$$\begin{aligned} P_{X+Y}(z) &= \int dx \int dy P_{X,Y}(x,y)\delta(x+y-z) \\ &= \int dx P_{X,Y}(x, z-x) \\ &= \int dx P_X(x) P_Y(z-x) \end{aligned}$$

Example 1. — Addition of two Uniform random variables.

Assume that

$x_1 \sim \mathrm{U}[0,1]$ **and** $x_2 \sim \mathrm{U}[0,1]$

$$P_Z(z) = \int_0^1 dx P_X(x) P_Y(z-x)$$

For $0 \le z \le 1$,

$$\begin{aligned} P_Y(z-x) &= 1 \quad \text{if } -1 \le x-z \le 0, \ \Rightarrow 0 \le x \le z \\ &= 0 \quad \text{if } x > z \end{aligned}$$

$$P_Z(z) = \int_0^z dx \cdot 1 = z$$

For $1 \le z \le 2$,

$$\begin{aligned} P_Y(z-x) &= 1 \quad \text{if } -1 \le x-z \le 0 \ \Rightarrow z-1 \le x \le 1 \\ &= 0 \quad \text{elsewhere} \end{aligned}$$

$$P_Z(z) = \int_{z-1}^1 dx \cdot 1 = 2-z$$

In summary,

$$\begin{aligned} P_z(z) &= z \qquad \text{if } 0 \le z \le 1 \\ &= 2-z \quad \text{if } 1 \le z \le 2 \end{aligned}$$

Example 2. — Addition of two Gamma random variables.

Assume that

$$x_1 \sim \mathrm{Gamma}(n+1,\lambda) \equiv \frac{\lambda^{n+1}x^n}{n!}\exp(-\lambda x) \text{ and}$$
$$x_2 \sim \mathrm{Gamma}(1,\lambda) \equiv \lambda\exp(-\lambda x),$$

where $x \geq 0$ and n is an integer greater than or equal to 0.

Then, the PDF of $z = x_1 + x_2$ is given by

$$\begin{aligned} P_Z(z) &= \int_0^z dx \frac{\lambda^{n+2}x^n}{n!}\exp(-\lambda x)\exp(-\lambda(z-x)) \\ &= \frac{\lambda^{n+2}}{n!}\exp(-\lambda z)\int_0^z dx \cdot x^n \\ &= \frac{\lambda^{n+2}z^{n+1}}{(n+1)!}\exp(-\lambda z) = \mathrm{Gamma}(n+2,\lambda) \end{aligned}$$

Note that the upper limit in the integral is z (not ∞) because

$$\begin{aligned} \mathrm{Gamma}(n,\lambda) &\equiv \frac{\lambda^n x^{n-1}}{(n-1)!}\exp(-\lambda x) \quad && \text{for } x \geq 0 \\ &= 0 && \text{otherwise} \end{aligned}$$

[Algorithm]

First, consider the simplest case:

$$x_1 \sim \mathrm{Gamma}(1,\lambda) = \lambda\exp(-\lambda x)$$

Draw a uniform random variable ξ_1 for this simplest case:

$$\xi = \int_0^x \lambda\exp(-\lambda x)dx = 1 - \exp(-\lambda x)$$
$$x = -\frac{1}{\lambda}\ln(1-\xi) \rightarrow x = -\frac{1}{\lambda}\ln(\xi)$$

In general, the random variables for a Gamma random variable Gamma(n, λ) can be obtained as follows:

Draw n uniform random numbers ξ_i ($i = 1,2,\cdots,n$).
Then, the desired random number is obtained by

$$\xi = -\frac{1}{\lambda}\left(\sum_{i=1}^{n}\ln\xi_i\right) = -\frac{1}{\lambda}\ln(\xi_1\xi_2\cdots\xi_n)$$

For example,

$$\xi = -\frac{1}{\lambda}\ln(\xi_1\xi_2\xi_3\xi_4) \text{ for } \xi \sim \frac{\lambda^4x^3}{3!}\exp(-\lambda x)$$

14. Sampling of frequencies for a blackbody radiation (Barnett-Canfield Algorithm)

Change of variable:
Introducing $x = h\nu / k_{\mathrm{B}}T$ (dimensionless frequency) reduces the Planck spectral density to:

$$b(x) = \frac{15}{\pi^4} \cdot \frac{x^3}{e^x - 1}, \; x \geq 0$$

The key identity:

$$\frac{1}{e^x - 1} = \frac{e^{-x}}{1 - e^{-x}} = \sum_{n=1}^{\infty} e^{-nx}$$

Substituting into the Planck function

$$b(x) = \frac{15}{\pi^4} \sum_{n=1}^{\infty} x^3 e^{-nx}$$

Each term is a proper Gamma(4, n) density (after normalizing):

$$f_n(x) = \frac{n^4}{6} \cdot x^3 e^{-nx} \quad \Leftarrow \quad \int_0^{\infty} x^n e^{-x} dx = n! = \Gamma(n+1)$$

Then,

$$b(x) = \sum_{n=1}^{\infty} \frac{90}{\pi^4} \frac{1}{n^4} f_n(x) = \sum_{n=1}^{\infty} w_n f_n(x)$$

Normalization check:

$$\sum_{n=1}^{\infty} w_n = \frac{90}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} = 1$$

Note that $\zeta(4) = \frac{\pi^4}{90}$

$$\int_0^{\infty} f_n(x) dx = \frac{1}{6} \int_0^{\infty} (nx)^3 e^{-nx} d(nx) = 1$$

Mixture Sampling Strategy:

(1) Sampling n from $w_n = \frac{90}{\pi^4} \frac{1}{n^4}$

(2) Sampling x from $f_n(x)$

or sampling y from $f(y) = \frac{1}{6} y^3 e^{-y}$ and $x = y/n$

(1) Samping n from w_n

Draw $\xi_0 \sim U[0,1]$.
Find smallest integer L such that:

$$\sum_{j=1}^{L} j^{-4} \geq \xi_0 \cdot \frac{\pi^4}{90}$$

Precomputed cumulative sums (note: $\pi^4/90 \approx 1.0823$)

L	$S_L = \Sigma j^{-4}$	S_L / (π^4/90)	w_L (%)
1	1.0000	0.9239	92.4
2	1.0625	0.9814	5.8
3	1.0748	0.9927	1.1
4	1.0787	0.9963	0.4
5	→ $\pi^4/90$	→ 1.0000	< 0.2

L = 1 is selected ~92% of the time — the loop rarely iterates past 2. Average: 1.1 iterations per sample (Fleck & Cummings 1971).

(2) Sampling x from $f_L(x)$

Draw three independent uniform random numbers $\xi_i \sim U[0,1]$ for $i = 1,\ 2,\ 3,\ 4$.

Then, the desired random number is obtained by

$$\xi = -\frac{1}{L}\ln(\xi_1\xi_2\xi_3\xi_4)$$

15. Simulating a Particle-Scattering Event

Algorithm

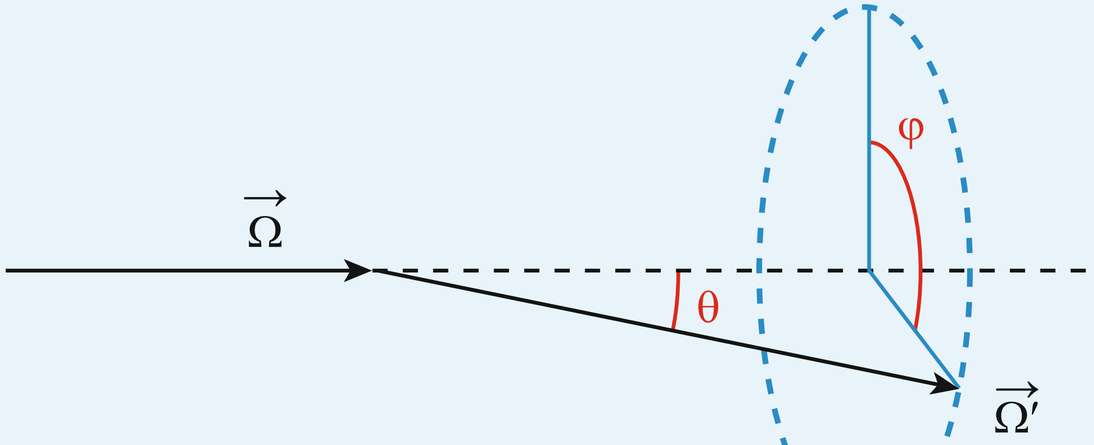
1. Sample E' (post-scatter energy)
2. Sample the scattering angle θ (polar angle)
3. Sample the azimuthal angle ϕ ~ Uniform[0, 2π]
4. Rotate direction: $\vec{\Omega} \to \vec{\Omega}'$

Direction rotation (general Ω):

$$\Omega_x' = \Omega_x \cos\theta - \frac{\sin\theta}{(1-\Omega_z^2)^{1/2}}\left(\Omega_x\Omega_z\cos\phi + \Omega_y\sin\phi\right)$$

$$\Omega_y' = \Omega_y \cos\theta - \frac{\sin\theta}{(1-\Omega_z^2)^{1/2}}\left(\Omega_y\Omega_z\cos\phi - \Omega_x\sin\phi\right)$$

$$\Omega_z' = \Omega_z \cos\theta + (1-\Omega_z^2)^{1/2}\sin\theta\sin\phi$$



θ = scattering angle

$\vec{\Omega}$ = particle direction before the collision

$\vec{\Omega}'$ = particle direction after the collision

Consider a system of coordinates with unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$:

$$\vec{\Omega} = \Omega_x\hat{\mathbf{x}} + \Omega_y\hat{\mathbf{y}} + \Omega_z\hat{\mathbf{z}}$$

$$\vec{\Omega}' = \Omega_x'\hat{\mathbf{x}} + \Omega_y'\hat{\mathbf{y}} + \Omega_z'\hat{\mathbf{z}}$$

Let's define a new system of coordinates $\hat{\mathbf{x}}'$, $\hat{\mathbf{y}}'$, and $\hat{\mathbf{z}}'$, where $\hat{\mathbf{z}}'$ is along the direction $\vec{\Omega}$ before scattering.

$$\hat{\mathbf{z}}' = \vec{\Omega}, \quad \hat{\mathbf{y}}' = \frac{\vec{\Omega}\times\hat{\mathbf{z}}}{\left|\vec{\Omega}\times\hat{\mathbf{z}}\right|}, \quad \text{and} \quad \hat{\mathbf{x}}' = \hat{\mathbf{y}}'\times\hat{\mathbf{z}}'$$

The new direction vector is expressed using the new system and scattering angles θ and ϕ:

$$\vec{\Omega}' = \sin\theta\cos\phi\hat{\mathbf{x}}' + \sin\theta\cos\phi\hat{\mathbf{y}}' + \cos\theta\hat{\mathbf{z}}'$$

Then, we express $\vec{\Omega}'$ in terms of ($\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$):

$$\Omega_x' = \hat{\mathbf{x}}\cdot\vec{\Omega}', \quad \Omega_y' = \hat{\mathbf{y}}\cdot\vec{\Omega}', \quad \text{and} \quad \Omega_z' = \hat{\mathbf{z}}\cdot\vec{\Omega}'$$

16. Algorithm Testing Methods

1

Histogram (Qualitative)

1. Generate large sample $\{\gamma_1, \gamma_2, \cdots, \gamma_N\}$
2. Build normalized histogram:

$$h_i = \frac{n_i}{N\Delta x_i}$$ = height of the bar

Δx_i = histogram bin width

n_i = number of random values that fell within the i-th bin.

$N = \sum_i n_i$ = sample size

The histogram is correctly normalized:

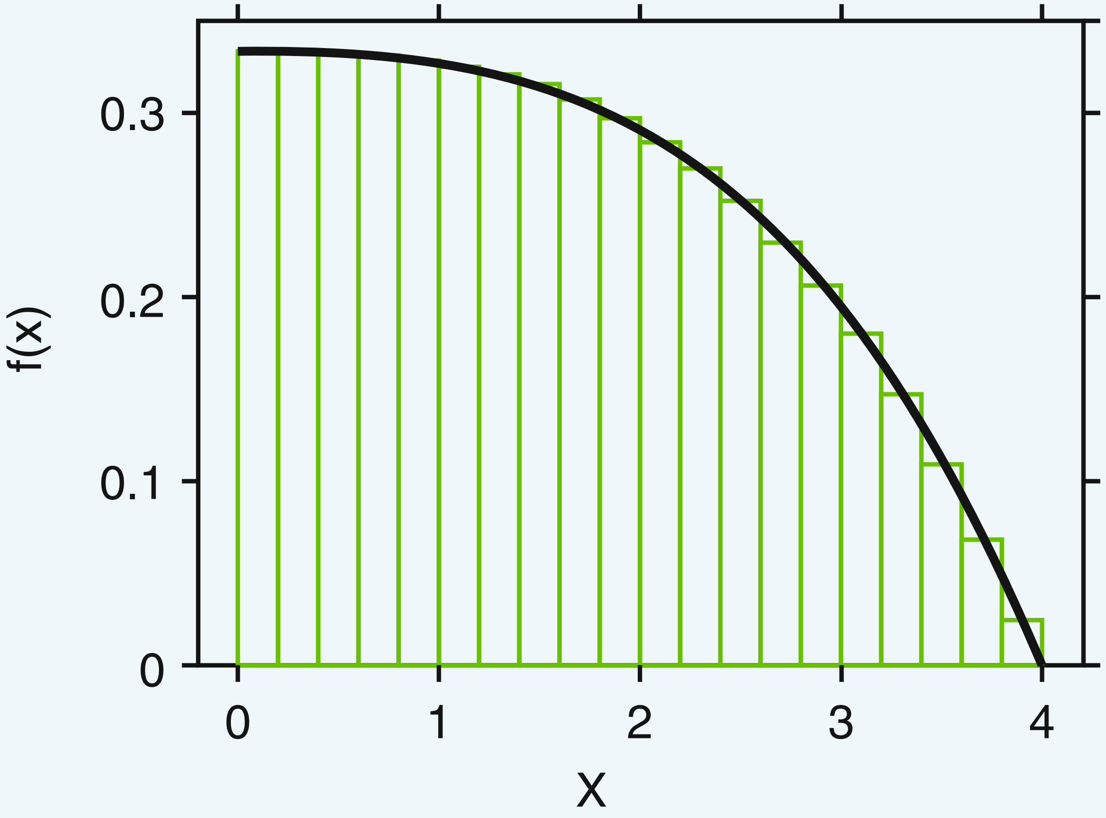
$$\sum_i h_i \Delta x_i = 1$$

3. Plot h_i vs $f(x)$

→ $f(x)$ should pass through the centers of the histogram bars at their tops.

⚠ Narrow bin at singularities

⚠ If $f(x)$ has a discontinuity, a bin boundary should be placed exactly at the point of discontinuity.



2

χ^2 Test (Quantitative)

1. Divide axis into k intervals
2. Expected probabilities: $p_i = \int_{a_i}^{b_i} f(x)dx$
3. Observed probabilities: $\pi_i = \frac{n_i}{N}$
4. Compute:

$$\chi^2(\text{obs}) = \sum_{i=1}^{k} \frac{(\pi_i - p_i)^2}{p_i}$$

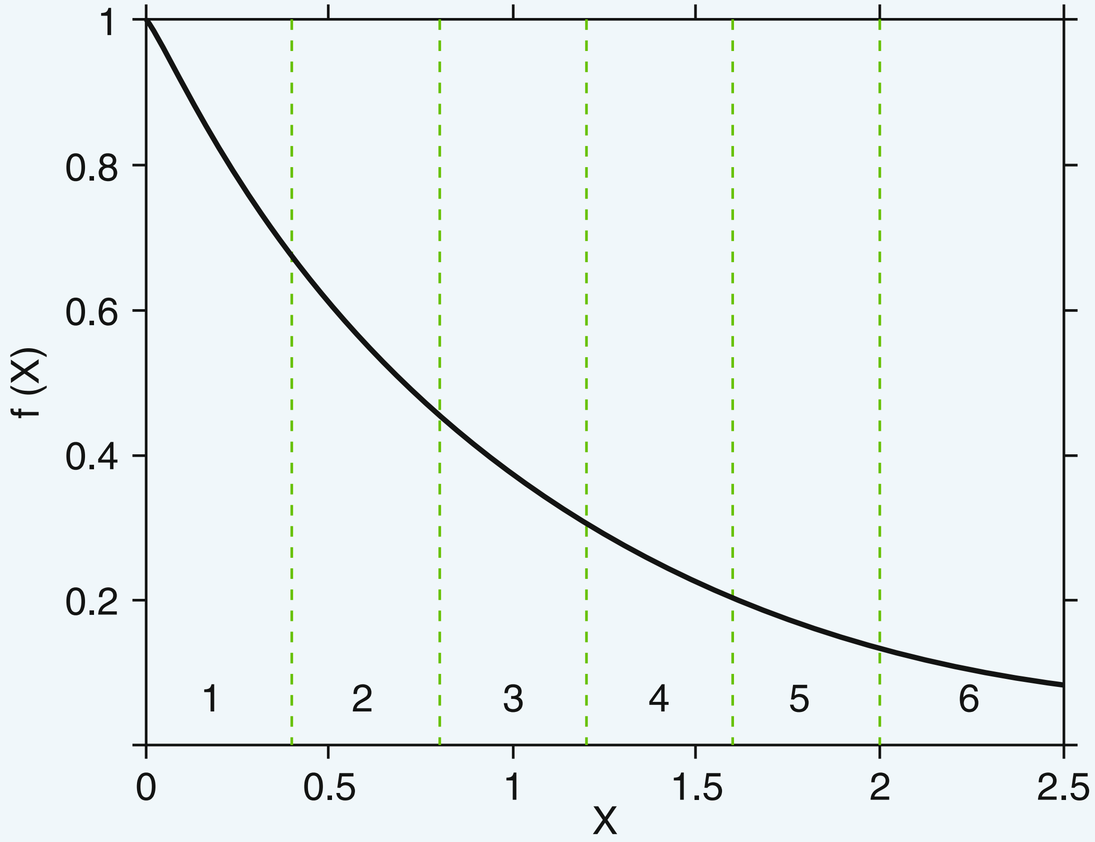
The sum is χ^2-distributed if each n_i is much greater than 1 and k is sufficiently large.

5. Using the χ^2 distribution, calculate the probability of observing a χ^2 that is equal or greater than χ^2(obs), using

$$P\{\chi^2 \geq \chi^2(\text{obs})\} = \int_{\chi^2(\text{obs})}^{\infty} p_{\chi^2}(x)dx$$

6. If the above probability is small, the hypothesis is rejected.

For example, $P(\chi^2 \geq \chi^2(\text{obs})) <$ 0.05 or 0.001.



3

Likelihood Ratio (Sensitive)

θ_0 = exact parameters
$\hat{\theta}$ = the maximum likelihood estimates of θ

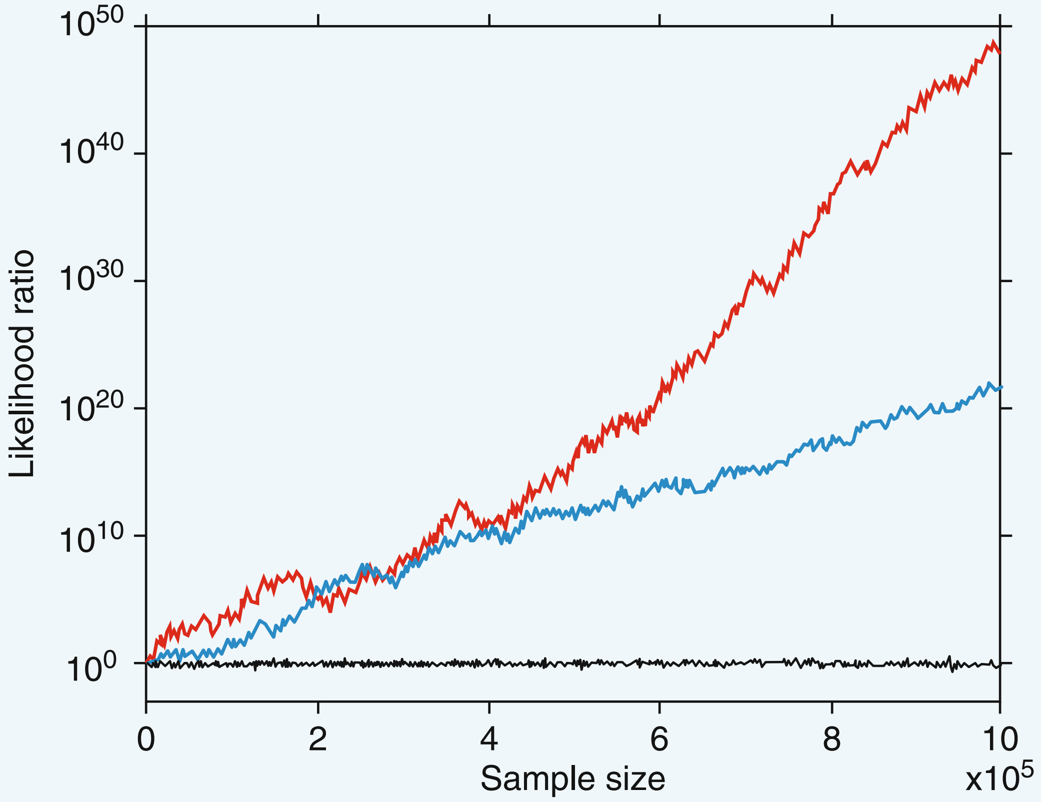
1. Generate a large sample $\{\xi_1, \xi_2, \cdots, \xi_N\}$.
2. Estimate parameters $\hat{\theta}$ from sample
3. Compute ratio:

$$\lambda_i = \prod_{j=1}^{i} \frac{f\left(\xi_j \mid \theta_0\right)}{f\left(\xi_j \mid \hat{\theta}\right)}; \quad i = 1,2,\cdots,N$$

4. Plot λ_i vs i

Correct code → λ_i will fluctuate around 1 (stable)
Error → λ_i diverges.

Very sensitive – detects even 1% errors in parameters!



Likelihood ratio test for a code sampling normal distribution N(1,1).

The black line shows the likelihood for a correct code.

The blue line indicates the code with a 1% error in the mean.

The red line represents a 1% error in the standard deviation.

Chapter 5: Monte Carlo Quanta

Overview of discretization paradigms

What are MC Quanta?

- MCRT discretizes radiation into a large number of packets.
- Each packet carries: position, propagation direction, frequency, energy/weight, and (optionally) a Stokes polarization vector.

5.1 Photon Packet Scheme

- Quanta = bundles of physical photons
- Weights encode photon count per packet
- Non-uniform & variable weights → variance reduction
- Dominant in dust RT simulations

5.2 Energy Packet Scheme

- Quanta = parcels of radiant energy
- Packet energy = its weight (conserved)
- Indivisible: no splitting during propagation
- Developed by L. Lucy (1999)

Key difference: photon packets conserve photon number; energy packets conserve energy.

5.1 Photon Packet Scheme

Historically dominant approach in astrophysical MCRT

Historical Inspiration

- Inspired by nature's own discretization.
- Early MCRT studies (Auer 1968; Avery & House 1968) referred to quanta simply as 'photons'.

Monte Carlo photon packets

- Each MC packet represents a vast number of real photons.
- The simulation count is completely insignificant compared to the actual radiation field.

Packet Weights

- Weights encode the physical photon count per packet.
- In this scheme, the MC quanta are typically referred to as ***photon packets*** or simply ***packets***. Non-uniform and variable weights reduce variance (MC noise) dramatically — a form of biasing (Sect. 9.4).

Applications

- Widely used in dust RT (Whitney 2011; Steinacker et al. 2013).
- Most astrophysical MCRT codes rely on this scheme with sophisticated weight manipulations.

5.2 Energy Packet Scheme

Lucy's indivisible energy packet formalism

Core Principle

- Packets are parcels of radiant energy.
- The packet's energy serves as its weight and is strictly conserved — packets are **indivisible** and **indestructible** (except at domain boundaries).

Indivisibility

- No splitting.
- Multiple outcome channels handled by probabilistic branching, preserving total energy throughout.

Energy Conservation

- Strict local energy conservation by construction in RE problems.
- Frequency can change at each interaction; CMF energy stays fixed.

Photon Number

- NOT conserved (physically correct: e.g. recombination cascades don't conserve photon number).
- A packet may represent varying photon counts during its lifetime.

- Many physical radiation-matter processes (e.g. recombination cascades or fluorescence) do not conserve the number of photons.

Key Advantages

- Effort proportional to energy carried, not photon count
- Avoids accumulation of negligible-weight quanta (no elimination scheme needed)
- Widely used: stellar winds, SN ejecta (Lucy 2005; Kerzendorf & Sim 2014)

Compton Scattering: Two Schemes Compared

A worked example illustrating the philosophical difference

Process: $e_i^- + \gamma_i \rightarrow e_f^- + \gamma_f$ *(photon transfers recoil energy to electron;* $\nu_f < \nu_i$*)*

Photon Packet Approach

- Packet scatters → photon number fixed
- Frequency reduced ($\nu_f < \nu_i$)
- Packet energy decreases accordingly

✓ Good for: computing Comptonized spectra

✗ Risk: many low-energy packets accumulate, each costing the same compute per scatter

Energy Packet Approach

- Track energy flow, not photon count
- $F_\gamma = E_f^\gamma / E_i^\gamma$ of the incident photon energy → is passed to the outgoing photon packet.
- $F_e = E_f^e / E_i^\gamma = 1 - F_\gamma$ → goes to the electron kinetic pool (*k*-packet)

✓ Energy strictly conserved

✓ Effort ∝ energy carried (ideal for heating applications)

Which is better?

- Neither is universally superior. Photon packets suit spectral calculations (Pozdnyakov et al. 1983).
- Energy packets suit heating applications (SN ejecta powered by radioactive decay, Lucy 2005).

5.3 Packet Initialization

Assigning initial properties via Monte Carlo sampling

Each packet is initialized with:

Position · Propagation direction · Frequency · Energy / Weight · Stokes vector (if polarization included)

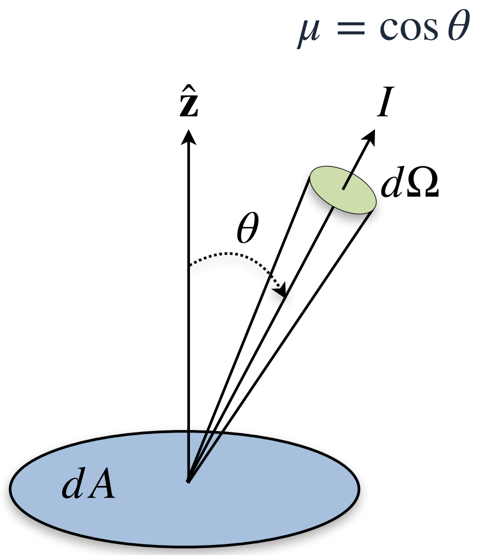
Case A: Uniform, Thermal Radiation from a volume ΔV

- Each of N packets carries: $\varepsilon = L = \Delta V \cdot a_R T^4 / N$
  ($L =$ luminosity, $a_R = (4/c)\sigma_{\rm SB}$)
- Position: uniform random in cuboid ΔV
  $x = x_0 + \xi_1(x_1 - x_0)$
  $y = y_0 + \xi_2(y_1 - y_0)$
  $z = z_0 + \xi_3(z_1 - z_0)$
- Direction: isotropic (Eq. 26-27): $\mu = 2\xi_1 - 1$ and $\phi = 2\pi\xi_2$
- Frequency: sampled from Planck function

Case B: Photospheric Boundary

- $N = 4\pi R_{\rm phot}^2 \sigma_{\rm SB} T_{\rm phot}^4 \Delta t / \varepsilon$
- Position: $r = R_{\rm phot}$ (on the photosphere)
- Direction: $\mu = \sqrt{\xi_1}$ and $\phi = 2\pi\xi_2$
  (flux-based; limb darkening neglected; isotropic radiation)
- Frequency: sampled from Planck function

- *Initialization is performed in the co-moving frame (CMF).*
- *Lab-frame (LF) properties are obtained via frame transformations (see Sect. 8 for expanding media).*



Contribution of I to the flux:

$P(\mu,\phi)\,d\mu\,d\phi\,dA = I \cdot d\mu\,d\phi\,(dA \cdot \mu)$

for $0 \le \mu \le 1$ $(0^\circ \le \theta \le 90^\circ)$

$\therefore P(\mu,\phi) = \mu\frac{1}{2\pi}$ after normalization

$\int_0^\mu d\mu' \mu' = \xi_1$ and $\int_0^\phi \frac{1}{2\pi} d\phi' = \xi_2$