외계행성 대기
Exoplanetary Atmospheres

Lecture 2: Intensity and Flux, Energy Balance, Temperatures, and Albedos
2026 September 14 (Monday)

Based on: Chapter 2 and Sections 3.1–3.4 of Seager, *Exoplanet Atmospheres* (2010)

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Outline – Lecture 2

Why spend a whole chapter on definitions?

The goal of the course is to understand the origin of a planetary spectrum and the physical processes that shape it: the gases and solid particles present, the planetary albedo, and the vertical temperature structure. Everything starts from a precise description of the *radiation* traveling through the atmosphere.

- The terms **intensity**, **surface flux**, and **flux at Earth** are used differently in other books and in the literature. We fix one consistent set of definitions and use it throughout (Seager's conventions).
- The same words hide subtleties: is the flux a vector or a scalar? Over which solid angle is it integrated? At the planet or at the detector? Is the "surface" a solid surface?
- In this chapter, **"surface"** means either the solid surface of a planet or the atmospheric layers from which the radiation emerges (the photosphere, in stellar language).

With these definitions in hand we can define planetary temperatures, albedos, and planet-star flux ratios (Chapter 3), and later write down the equation of radiative transfer (Chapter 5).

The specific intensity I

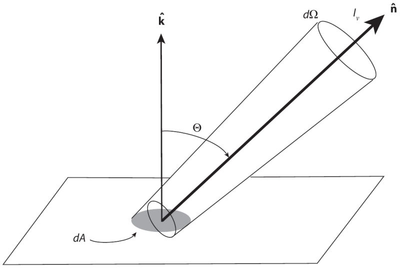

Figure 1: Definition of the specific intensity $I(\mathbf{x}, \hat{\mathbf{n}}, \nu, t)$. Reproduced from Seager (2010), Fig. 2.1.

Radiation is energy carried by photons. The conventional description considers the energy of a number of identical photons in a single *beam*: the intensity I. It is the energy passing through an area $\mathrm{d}A$, within a solid angle $\mathrm{d}\Omega$ centered on the direction $\hat{\mathbf{n}}$, per frequency interval, per unit time:

$$\boxed{\mathrm{d}E(\nu,t) = I(\mathbf{x}, \hat{\mathbf{n}}, \nu, t)\ \hat{\mathbf{n}} \cdot \hat{\mathbf{k}}\ \mathrm{d}\Omega\, \mathrm{d}A\, \mathrm{d}\nu\, \mathrm{d}t}$$

- SI units: $\mathrm{J\,m^{-2}\,sr^{-1}\,s^{-1}\,Hz^{-1}}$.
- $\hat{\mathbf{k}}$ is the normal to $\mathrm{d}A$; the factor $\hat{\mathbf{n}} \cdot \hat{\mathbf{k}} = \cos\Theta$ is the *projected* area seen by the beam.
- I is a *scalar* that carries a direction: intensity at location $\mathbf{x}$ going into direction $\hat{\mathbf{n}}$.
- I is macroscopic: it sums all microscopic absorption and emission processes acting on the beam.

Why we must carry I through the calculation

- As a beam travels through the atmosphere, photons are absorbed from and emitted into it. Different parts of the planet therefore have different intensities, and I varies with frequency.
- We can *never* measure I from a specific part of an exoplanet: the planet is spatially unresolved. What we measure is a *flux*, an integral of I over the visible hemisphere.
- Nevertheless, radiative transfer is formulated in terms of I. We compute I in detail everywhere in the atmosphere and only at the end integrate to the observable quantities.

Two properties worth remembering (exercises 2.2 and 2.3):

- In empty space (no extinction, no emission), I is *constant along a ray*: the energy through $\mathrm{d}A_1\mathrm{d}\Omega_1$ equals that through $\mathrm{d}A_2\mathrm{d}\Omega_2$ because $\mathrm{d}A_1\mathrm{d}\Omega_1 = \mathrm{d}A_2\mathrm{d}\Omega_2$ (the étendue is conserved). Intensity does not fall off with distance.
- The *flux* of a source does fall off as $1/d^2$: the solid angle subtended by the source shrinks as $1/d^2$ while I stays fixed. There is no contradiction.

Solid angle and the mean intensity

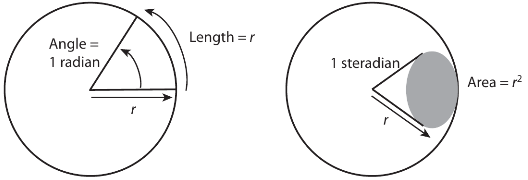

Figure 2: A radian (left) and a steradian (right). In 2D, 1 radian is the angle subtended at the center of a circle by an arc equal to the radius; in 3D, 1 steradian is the solid angle subtended at the center of a sphere of radius r by an area r^2. Reproduced from Seager (2010), Fig. 2.2.

The full sphere subtends 4π steradians. A cone of half-opening angle θ subtends (exercise 2.4)

$$\Omega = \frac{1}{R^2}\int_0^{\theta} 2\pi R\sin\theta' \, R\,\mathrm{d}\theta' = 2\pi(1-\cos\theta),$$

which tends to 4π as $\theta \to \pi$ and to $\pi\theta^2$ (the area of a small disk over R^2) as $\theta \to 0$.

The **mean intensity** is the intensity averaged over all solid angles, the *zeroth moment* of I:

$$\boxed{J(\mathbf{x},\nu,t) = \frac{1}{4\pi}\oint_{\Omega} I(\mathbf{x},\hat{\mathbf{n}},\nu,t)\,\mathrm{d}\Omega}$$

For isotropic radiation $J = I$. J is what an absorbing particle "feels": the heating rate of a gas parcel depends on J, not on the flux.

Flux: the first moment of the intensity

The quantity we actually measure from exoplanets is related to the **flux**: the *net* flow of energy through an arbitrarily oriented area $\mathrm{d}A$ with normal $\hat{\mathbf{n}}$, per frequency interval, per unit time. As a vector,

$$\mathbf{F}(\mathbf{x},\nu,t) = \oint_{\Omega} I(\mathbf{x},\hat{\mathbf{n}},\nu,t)\,\hat{\mathbf{n}}\,\mathrm{d}\Omega, \qquad [\mathbf{F}] = \mathrm{J\,m^{-2}\,s^{-1}\,Hz^{-1}},$$

defined at each point $\mathbf{x}$ of the atmosphere. In Cartesian components, $\mathbf{F} = F_i\,\hat{\mathbf{i}} + F_j\,\hat{\mathbf{j}} + F_k\,\hat{\mathbf{k}}$.

- In planetary atmospheres we usually want the flux in *one* direction, toward the observer. We take one component of the vector and write only its magnitude, a scalar:

$$\boxed{F_k(\mathbf{x},\nu,t) = \frac{\mathrm{d}E(\nu,t)}{\mathrm{d}A\,\mathrm{d}\nu\,\mathrm{d}t} = \oint_{\Omega} I(\mathbf{x},\hat{\mathbf{n}},\nu,t)\,\hat{\mathbf{n}}\cdot\hat{\mathbf{k}}\,\mathrm{d}\Omega}$$

- The directional subscript is then usually dropped: $F(\mathbf{x},\nu,t)$ means the flux in the direction of the observer.
- Since $\hat{\mathbf{n}}\cdot\hat{\mathbf{k}}$ changes sign across the plane of $\mathrm{d}A$, F is a *net* flux: energy flowing forward minus energy flowing backward. For *isotropic* radiation the net flux through any surface vanishes, $F = 0$, although the outward flux through one hemisphere is πI (exercise 2.2).

Higher moments: K and the radiation pressure

The *second moment* of the intensity is the tensor

$$\mathbf{K}(\mathbf{x},\nu,t) = \frac{1}{4\pi}\oint_\Omega I(\mathbf{x},\hat{\mathbf{n}},\nu,t)\,\hat{\mathbf{n}}\hat{\mathbf{n}}\,\mathrm{d}\Omega,$$

related to the **radiation pressure tensor** by $\mathbf{P} = \frac{4\pi}{c}\mathbf{K}$, i.e. $P_{ij} = \frac{1}{c}\oint I\,n_i n_j\,\mathrm{d}\Omega$. When K is used (Chapter 6, the Eddington approximation), only the magnitude of the component of interest is kept, a scalar, exactly as for the flux.

The hierarchy of moments (in a plane-parallel atmosphere with $\mu = \cos\theta$, Chapter 5):

$$J = \frac{1}{2}\int_{-1}^{1} I\,\mathrm{d}\mu, \qquad H = \frac{1}{2}\int_{-1}^{1}\mu\,I\,\mathrm{d}\mu = \frac{F}{4\pi}, \qquad K = \frac{1}{2}\int_{-1}^{1}\mu^2 I\,\mathrm{d}\mu.$$

- J: energy density ($u = 4\pi J/c$) and heating; H (or F): energy transport; K: radiation pressure ($P = 4\pi K/c$).
- For isotropic radiation $K = J/3$ and $P = u/3$, the familiar radiation-pressure relation.
- The stellar-atmosphere literature often uses the “astrophysical flux” $H = F/4\pi$ (or $\mathscr{F} = F/\pi$) because its form parallels J and K. Watch the factors of π when comparing books.

Surface flux F_S: flux emerging from the planet

For exoplanet atmospheres we are most interested in the flux emerging from the planet "surface", the layers where most of the radiation originates without further interaction with gas, liquid, or solid particles. We call it the **surface flux** F_S and distinguish it from F, which can be defined in any small volume inside the atmosphere.

- F_S is the *outgoing* flux at a particular point on the surface: the magnitude of the flux vector in the outgoing direction of interest, a scalar.
- To write it down we need a coordinate system for the surface point, *and* a coordinate system for the directions $\hat{\mathbf{n}}$ of the rays leaving that point. Two choices are convenient: latitude–longitude (next slide) and colatitude (the slide after).

Coordinates on the planet. In the spherical polar system (r, θ, ϕ), θ is the latitude and ϕ the longitude. For the planet–star problem the "equator" $\theta = 0$ is the orbital (ecliptic) plane, so $(\theta, \phi) = (0, 0)$ is the **substellar point**, the point that receives the most stellar radiation. (On Earth the equator is instead defined by the rotation axis.) Because we want only the surface intensity, we drop r and set $r = R_p$.

Latitude–longitude coordinates and $\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}$

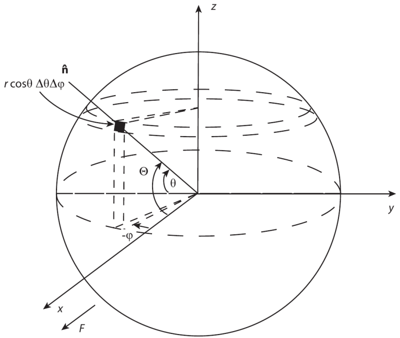

Figure 3: Spherical polar coordinates with latitude θ and longitude ϕ. For flux along the x-axis ($\hat{\mathbf{k}} = \hat{x}$), $\hat{\mathbf{n}} \cdot \hat{\mathbf{k}} = \cos\Theta$. Reproduced from Seager (2010), Fig. 2.3.

The intensity now has two sets of angles: (θ, ϕ) for the surface point and (θ_n, ϕ_n) for the ray direction $\hat{\mathbf{n}}$, so $I_S(\theta, \phi, \theta_n, \phi_n, \nu, t)$.

- With the flux direction $\hat{\mathbf{k}}$ along the x-axis (toward the observer), the spherical cosine law gives

$$\hat{\mathbf{n}} \cdot \hat{\mathbf{k}} = \cos\Theta_n = \cos\theta_n \cos\phi_n$$

  (exercise 2.5): a ray tilted in latitude *and* in longitude has its projection reduced by both cosines.
- The solid-angle element is $\mathrm{d}\Omega_n = \cos\theta_n \, \mathrm{d}\theta_n \, \mathrm{d}\phi_n$ (the $\cos\theta_n$ is the Jacobian of latitude, since latitude is measured from the equator, not from the pole).
- Using θ_n, ϕ_n we no longer need $\hat{\mathbf{n}}$ or $\hat{\mathbf{k}}$ explicitly in the description of the intensity.

Colatitude coordinates: the symmetric choice

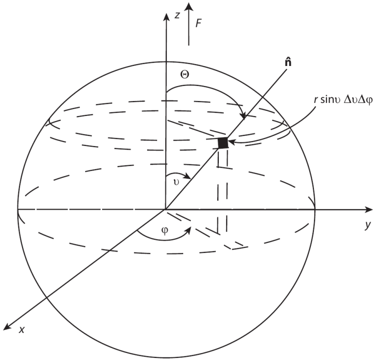

Figure 4: Spherical polar coordinates with colatitude ϑ and longitude ϕ. For flux along the z-axis, $\hat{\mathbf{n}} \cdot \hat{\mathbf{k}} = \cos\vartheta$. Reproduced from Seager (2010), Fig. 2.4.

Often it is easier to put one angle at the z-axis: use the **colatitude** $\vartheta = 90^\circ - \theta$ instead of the latitude. The substellar point is now at the pole and the flux is specified along z. The benefit:

$$\hat{\mathbf{n}} \cdot \hat{\mathbf{k}} = \cos\Theta_n = \cos\vartheta_n, \qquad d\Omega_n = \sin\vartheta_n \, d\vartheta_n \, d\phi_n.$$

- Only one angle enters the projection factor.
- For problems symmetric in ϕ (a star at the zenith of the substellar point, an isotropically emitting surface) the azimuthal integral is trivial and gives 2π.
- This is the system used to derive the flux observed at Earth and the incident stellar flux below, with the z-axis pointing at the observer or at the star.

The surface flux in the two coordinate systems

Inserting the projection factors and solid-angle elements into the flux definition, the surface flux at the point (θ, ϕ) or (ϑ, ϕ) is

$$F_{\mathrm{S}}(\theta, \phi, \nu, t) = \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} I_{\mathrm{S}}(\theta, \phi, \theta_n, \phi_n, \nu, t) \cos\phi_n \cos^2\theta_n \, \mathrm{d}\theta_n \, \mathrm{d}\phi_n,$$

$$F_{\mathrm{S}}(\vartheta, \phi, \nu, t) = \int_0^{2\pi} \int_0^{\pi/2} I_{\mathrm{S}}(\vartheta, \phi, \vartheta_n, \phi_n, \nu, t) \cos\vartheta_n \sin\vartheta_n \, \mathrm{d}\vartheta_n \, \mathrm{d}\phi_n.$$

- The limits ($\pm\pi/2$ in latitude–longitude; 0 to $\pi/2$ in colatitude) select the *outgoing* hemisphere. In which direction does the flux travel? Out from the planet along $\hat{\mathbf{k}}$, at the location (θ, ϕ).
- **Common shorthand.** The surface flux is usually written without reference to the surface point,

$$F_{\mathrm{S}}(\nu, t) = \int_0^{2\pi} \int_0^{\pi/2} I_{\mathrm{S}}(\vartheta, \phi, \nu, t) \cos\vartheta \sin\vartheta \, \mathrm{d}\vartheta \, \mathrm{d}\phi,$$

  where ϑ, ϕ now refer to the *directions* of the rays and the surface element is left unspecified. Less precise, but adequate for a planet with uniform surface intensity, and we will “be guilty of adopting it”.

The basic result: $F_S = \pi I_S$ for uniform intensity

For a planet with uniform (isotropic) surface intensity $I_S(\theta,\phi,\nu,t) = I_S(\nu,t)$, the intensity comes out of the integrand and the angular integral over the outgoing hemisphere gives

$$\boxed{F_S(\nu,t) = \pi I_S(\nu,t)} \qquad \text{since} \qquad \int_0^{2\pi}\int_0^{\pi/2} \cos\vartheta \sin\vartheta \, d\vartheta \, d\phi = 2\pi \cdot \tfrac{1}{2} = \pi.$$

- The factor is π, not 2π: rays leaving at grazing angles carry little energy *through* the surface because of the projection factor $\cos\vartheta$.
- This is why the blackbody *flux* is πB while the blackbody *intensity* is B.
- The same integral in latitude–longitude form, $\iint \cos\phi \cos^2\theta \, d\theta \, d\phi = \pi$, gives the same answer, as it must.

Frequency-integrated (total) flux. When we integrate over all frequencies we drop the ν argument:

$$F_S(t) = \int_0^\infty F_S(\nu,t)\, d\nu \qquad [\mathrm{J\,m^{-2}\,s^{-1}}].$$

Observed flux: what we measure at Earth

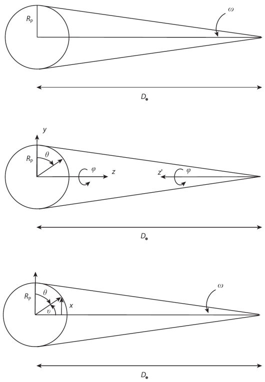

Figure 5: Derivation of the planet flux observed at Earth. Seager (2010), Fig. 2.5.

What form of radiation are we able to measure at Earth? At Earth we see the planet as a point source, spatially unresolved: all radiation from the exoplanet hemisphere is averaged into a single value of flux, $F_\oplus(\nu, t)$.

- Recall that the surface intensity depends on the location on the planet, described by θ and ϕ. We must integrate the surface intensity from each location into a quantity we can actually measure.
- At Earth we measure the energy collected from a planet subtending an angle Ω, by a detector of a given area, in a frequency interval, during some interval of time.
- **The key point:** to derive $F_\oplus$ for a distant planet we must realize that the integration over solid angle is done *at the detector*, not at the planet surface.
- We return to the definition of flux and use the colatitude coordinate system, which is convenient here: the flux leaves the planet along the z-axis, and the z-axis points toward the detector at Earth. The angle ϕ is defined about the z-axis.

Observed flux: the solid angle subtended by the planet

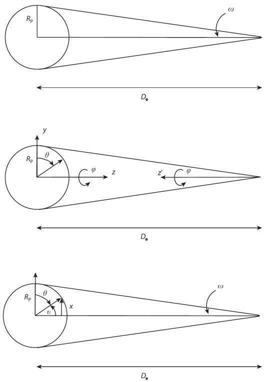

Figure 6: Top: the angle $R_p/D_\oplus$. Middle: the shared azimuth ϕ. Bottom: ω versus ϑ. Seager (2010), Fig. 2.5.

- From the detector at Earth, the planet subtends the solid angle

$$\Omega = \int_0^{2\pi}\int_0^{R_p/D_\oplus} \sin\omega \, \mathrm{d}\omega \, \mathrm{d}\phi \simeq \pi\left(\frac{R_p}{D_\oplus}\right)^2,$$

  where ω is the angle measured from the planet's center as seen from Earth (top panel), running from 0 at the disk center to $R_p/D_\oplus$ at the limb.

- A critical point: the ϕ component of the angle subtended by the planet at Earth is *equivalent* to the angle ϕ in the colatitude coordinate system on the planet (middle panel). The azimuth about the line of sight is the same whether measured at the planet or at the detector.

- Each point of the planet's disk, at angular position (ω, ϕ) as seen from Earth, corresponds to one surface element at colatitude ϑ on the planet (bottom panel); the ray from that element to Earth leaves the surface at angle ϑ from the local normal.

Observed flux: the integral at the detector

In the coordinate system of the detector, the flux is

$$F_\oplus(\nu,t) = \int_0^{2\pi}\int_0^{R_p/D_\oplus} I_S(\vartheta,\phi,\nu,t)\cos\omega\sin\omega\,\mathrm{d}\omega\,\mathrm{d}\phi,$$

where the intensity is still described in the planet's coordinates rather than in those of the detector. We now convert $\cos\omega\sin\omega\,\mathrm{d}\omega$ into the planet's coordinates. From the bottom panel of Fig. 2.5, a ray leaving the surface point at colatitude ϑ toward Earth appears at the angle

$$\omega = \frac{R_p}{D_\oplus}\sin\vartheta \qquad \Longrightarrow \qquad \mathrm{d}\omega = \frac{R_p}{D_\oplus}\cos\vartheta\,\mathrm{d}\vartheta.$$

- In the limit $R_p \ll D_\oplus$ we have $\omega \ll 1$, so by Taylor expansion $\sin\omega \simeq \omega$ and $\cos\omega \simeq 1-\omega^2/2 \simeq 1$.
- In the same limit $D_\oplus$ can be taken as the distance from the surface of the star to the observer at Earth.

The integral then becomes

$$F_\oplus(\nu,t) = \left(\frac{R_p}{D_\oplus}\right)^2 \int_0^{2\pi}\int_0^{\pi/2} I_S(\vartheta,\phi,\nu,t)\cos\vartheta\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\phi,$$

in which the angular integral is exactly the surface-flux integral.

Observed flux: the result

$$\boxed{F_{\oplus}(\nu,t) = \left(\frac{R_p}{D_{\oplus}}\right)^2 F_{\mathrm{S}}(\nu,t)}$$

- $F_{\oplus}$ has the same dimensions as F_{S} and is smaller by the dilution factor $(R_p/D_{\oplus})^2$: the inverse-square law, now derived rather than assumed. For a Jupiter at 10 pc, $(R_p/D_{\oplus})^2 = 5 \times 10^{-17}$.
- The result is exact only for a planet with uniform surface intensity and $R_p \ll D_{\oplus}$. For a nonuniform planet, $F_{\oplus}$ is a hemisphere-weighted average with the $\cos\vartheta$ weighting: regions near the limb contribute less than regions near the center of the disk. This weighting is what phase curves and eclipse maps exploit to recover the brightness distribution.
- The surface flux and the observed flux are thus the "same" quantity up to a geometric factor, provided the planet is far away and its intensity is uniform. This is the justification for talking about the planet's "flux" without specifying where it is evaluated, which we will often do.
- The same derivation with the roles reversed gives the stellar flux incident on the planet (next section).

Luminosity and outgoing energy

Luminosity $L(t)$ is the rate at which a planet radiates energy in *all* directions: the surface flux summed over a closed surface around the planet. For uniform flux $F_\mathrm{S}(t)$ radiated equally everywhere, with $\mathrm{d}A = R_p^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$,

$$L(t) = \oint_A F_\mathrm{S}(t) \, \mathrm{d}A = R_p^2 \int_0^{2\pi} \int_0^{\pi} F_\mathrm{S}(t) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \quad \Longrightarrow \quad \boxed{L(t) = 4\pi R_p^2 F_\mathrm{S}(t)} \quad [\mathrm{J\,s^{-1}}].$$

- For giant planets, L is also useful for the flux from the *interior* incident on the *lower* boundary of the atmosphere: uniform around the planet and coming from the gravitational potential energy of the planet's contraction.
- **Caution:** when observing, only *one hemisphere* is visible. For a planet that does not radiate uniformly (a hot dayside and a cold nightside) we must not use L; instead use the energy per unit time leaving one hemisphere,

$$E_\mathrm{S}(t) = R_p^2 \int_0^{\pi} \int_0^{\pi/2} F_\mathrm{S}(\theta, \phi, t) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \xrightarrow{\text{uniform}} E_\mathrm{S}(t) = 2\pi R_p^2 F_\mathrm{S}(t).$$

Energy per unit time is technically *power*; the planetary literature says "energy per unit time" or "luminosity".

Incident flux: the star as seen from the planet

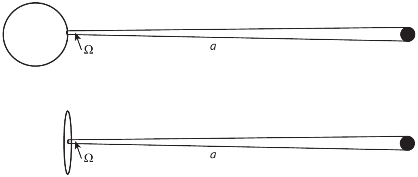

Figure 7: Incident flux on a sphere and on a disk. The star subtends a solid angle Ω as seen from each location on the planet; a is the semimajor axis. Reproduced from Seager (2010), Fig. 2.6.

The stellar radiation heats the planet, governs its global energy balance, and drives the mass motions of the atmosphere. We call it incident radiation, incident flux, or *irradiation*.

Reuse the result for the planet flux at Earth, with the roles swapped: replace $R_p \to R_*$, $D_\oplus \to a$, and $I_\mathrm{S} \to I_{\mathrm{S},*}$. The flux from the star at a point on the planet is

$$F_\mathrm{inc}(\nu,t) = \left(\frac{R_*}{a}\right)^2 \int_0^{2\pi}\int_0^{\pi/2} I_{\mathrm{S},*} \cos\vartheta \sin\vartheta \, \mathrm{d}\vartheta \, \mathrm{d}\phi \, .$$

For a star of uniform surface intensity (no limb darkening),

$$\boxed{F_\mathrm{inc}(\nu,t) = \left(\frac{R_*}{a}\right)^2 \pi I_{\mathrm{S},*}(\nu,t) = \left(\frac{R_*}{a}\right)^2 F_{\mathrm{S},*}(\nu,t)}$$

This is the flux at the **substellar point**, where the star is at the zenith.

Incident energy: integrating over the dayside

The total incident energy per unit time is *not* $2\pi R_p^2 F_{\rm inc}$: only the substellar point receives the full flux. A point at angle Θ from the substellar point receives $F_{\rm inc}\cos\Theta$ because the surface element is tilted (this is why Earth's poles are colder than the equator). Integrating over the illuminated hemisphere,

$$E_{\rm inc}(\nu,t) = \left(\frac{R_*}{a}\right)^2 F_{{\rm S},*}(\nu,t)\int_0^{2\pi}\int_0^{\pi/2} R_p^2 \cos\vartheta \sin\vartheta \, {\rm d}\vartheta \, {\rm d}\phi \quad \Longrightarrow \quad \boxed{E_{\rm inc}(\nu,t) = \pi R_p^2 \left(\frac{R_*}{a}\right)^2 F_{{\rm S},*}(\nu,t)}$$

and, integrated over all wavelengths, $E_{\rm inc}(t) = \pi R_p^2 (R_*/a)^2 F_{{\rm S},*}(t)$.

- The $\cos\vartheta$ integral turns the hemisphere ($2\pi R_p^2$) into the **cross-sectional disk** πR_p^2: the planet intercepts starlight with its projected area. This is the "disk" in Fig. 2.6.
- Sanity check with $L_* = 4\pi R_*^2 F_{{\rm S},*}$: the stellar flux at distance a is $L_*/4\pi a^2 = (R_*/a)^2 F_{{\rm S},*}$, and the planet intercepts πR_p^2 of it. For Earth, $L_\odot/4\pi a^2 = 1361\,{\rm W\,m^{-2}}$ (the solar constant).

The star as a point source: plane-parallel incident radiation

For radiative-transfer calculations at a given surface element, the star is far enough away that only rays from *one* direction are incident. We then describe the incident stellar intensity with delta functions,

$$I_*(\vartheta,\phi,\nu,t) = I_0\,\delta(\vartheta-\vartheta_0)\,\delta(\phi-\phi_0),$$

where (ϑ_0,ϕ_0) is the direction of the star from the local surface normal; it is different for every surface element. The incident flux at the element is then

$$F_{\rm inc}(\vartheta,\phi,\nu,t) = \left(\frac{R_*}{a}\right)^2 \int_0^{2\pi}\int_0^{\pi/2} I_0\,\delta(\vartheta-\vartheta_0)\delta(\phi-\phi_0)\cos\vartheta\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\phi = \left(\frac{R_*}{a}\right)^2 \cos\vartheta_0\sin\vartheta_0\,I_0.$$

- The $\cos\vartheta_0$ is the projection factor; $\sin\vartheta_0$ comes from the solid-angle Jacobian and the normalization of the delta functions in (ϑ,ϕ).
- This "plane-parallel beam" form is the upper boundary condition of the plane-parallel radiative transfer equation (Chapter 5): $I(0,\mu,\nu) = I_*\,\delta(\mu-\mu_0)$ for incoming rays.
- *When is the point-source approximation valid?* (exercise 2.7) When the star's angular radius R_*/a is small compared to the angular scales over which the atmosphere's response varies. For a hot Jupiter at $a = 0.05$ AU, $R_*/a \simeq 0.09\,\mathrm{rad} \simeq 5^\circ$, so near the terminator the finite size of the star (and its limb darkening) already matters.

Blackbody intensity and blackbody flux

A **blackbody** is a perfect radiator: it absorbs all incident radiation and reemits with a spectrum that depends only on its temperature T. Blackbody radiation is *isotropic* (no $\hat{\mathbf{n}}$-dependence). Its intensity is the **Planck function**

$$\boxed{B(T,\nu) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/k_B T} - 1}} \qquad [\mathrm{J\,m^{-2}\,sr^{-1}\,s^{-1}\,Hz^{-1}}],$$

the same dimensions as I. Since T varies with location (and possibly time) we may also write $B(\mathbf{x},\nu)$.

- **Per wavelength.** Blackbody radiation per wavelength bin $d\lambda$ requires the conversion $d\nu = -(c/\lambda^2)d\lambda$ (the sign is absorbed into the direction of $d\lambda$):

$$B(T,\lambda) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda k_B T} - 1}, \qquad B_\lambda\, d\lambda = B_\nu\, d\nu.$$

  B_λ and B_ν peak at *different* places (next slide). A classic novice pitfall (Heng, problem 2.8.4).

- **Blackbody flux.** Because B is isotropic, the flux outward through one hemisphere is

$$\boxed{F_S(T,\nu) = \pi B(T,\nu)}$$

  We always use B for blackbody intensity and πB for blackbody flux.

Blackbody fluxes across the relevant temperatures

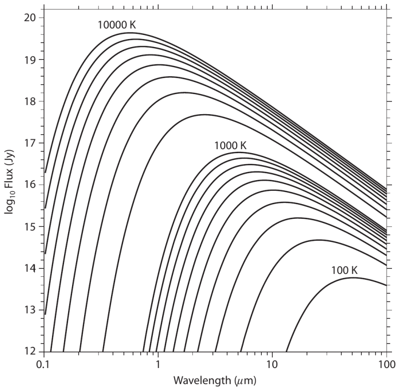

Figure 8: Blackbody surface fluxes πB (in Jy $= 10^{-26}$ W m^{-2} Hz^{-1}), spaced by 100 K from 100 to 1000 K and by 1000 K from 1000 to 10,000 K. Seager (2010), Fig. 2.7.

- Stars with known planets have $T_{\rm eff} \simeq 3000$–6000 K (the Sun may be represented by a 5750 K blackbody). Known planets have $T_{\rm eff} \simeq 60$ to more than 2000 K (Earth is approximately a 300 K blackbody).
- Both the *magnitude* and the *frequency of the peak* of the blackbody flux increase with temperature; the curves never cross. A hotter body is brighter at *every* wavelength.
- Stellar and planetary atmospheres can be *approximated* as blackbodies, but a blackbody is a highly idealized construct; the actual spectra depart from it in bands and lines, which is where the information about composition lies.

The two sides of the Planck function

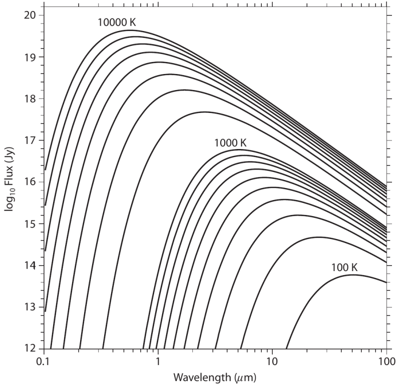

Figure 9: Same as the previous slide; note the common slope on the long-wavelength side. Seager (2010), Fig. 2.7.

- **Rayleigh–Jeans side** ($h\nu \ll k_B T$, $\lambda \gg \lambda_{max}$): $B \simeq 2\nu^2 k_B T/c^2$. On this logarithmic plot it is a straight line of slope -2 in λ (slope $+2$ in ν), with the temperature entering only *linearly*. This is why the mid-infrared planet-to-star contrast is $\propto T_p/T_*$ rather than exponentially small (Lecture 1, and next lecture).
- **Wien side** ($h\nu \gg k_B T$): $B \simeq (2h\nu^3/c^2)e^{-h\nu/k_B T}$, exponentially sensitive to T. A planet's flux at wavelengths shortward of its peak is negligible, which is why planets are invisible in the optical except by reflected light.
- The **peak** moves as $\lambda_{max} \propto 1/T$ (Wien's law, next slide); the **total** flux grows as T^4 (Stefan–Boltzmann, Section 3.3).

Wien's law: where does a blackbody peak?

Setting $\partial B(T,\lambda)/\partial\lambda = 0$ or $\partial B(T,\nu)/\partial\nu = 0$ (exercise 2.6) gives **Wien's displacement law** in the two forms

$$\boxed{\lambda_{\max} T = 2898\ \mu\mathrm{m\,K}} \qquad \boxed{\frac{\nu_{\max}}{T} = 5.88\times10^{10}\ \mathrm{Hz\,K^{-1}}}$$

The two peaks are *not* at the same place: $\lambda_{\max}\nu_{\max} \neq c$, because B_λ and B_ν are densities per unit wavelength and per unit frequency, and $\mathrm{d}\nu/\mathrm{d}\lambda = -c/\lambda^2$ redistributes the weight. (Wien derived the law from thermodynamics; the numerical constants come from the transcendental equations $x = 5(1-\mathrm{e}^{-x})$ and $x = 3(1-\mathrm{e}^{-x})$, with $x = hc/\lambda k_{\mathrm{B}}T$ or $h\nu/k_{\mathrm{B}}T$.)

object	T (K)	$\lambda_{\max}$ of B_λ	$c/\nu_{\max}$ of B_ν
Sun (G2)	5750	0.50 μm	0.88 μm
M star	3500	0.83 μm	1.46 μm
hot Jupiter	1500	1.9 μm	3.4 μm
Earth	300	9.7 μm	17 μm
Jupiter	125	23 μm	41 μm

Consequences of Wien's law

- The Sun's B_λ peaks in the green; its B_ν peaks in the near-infrared. Neither is "the" color of the Sun; the choice of variable is a convention. When a paper quotes "the peak of the spectrum", check which variable is meant.
- Planets radiate in the thermal infrared (5–50 μm), stars in the visible and near-infrared: the two components of a planet's spectrum, reflected starlight and thermal emission, are well separated in wavelength, except for the hottest planets ($T \gtrsim 2000$ K), whose thermal emission reaches into the near-infrared and even the red optical.
- Detecting a planet's thermal emission means observing near its Wien peak or beyond: 3–10 μm for hot Jupiters (Spitzer, JWST), 10–25 μm for temperate terrestrial planets (the 10 μm window for Earth analogs).
- Conversely, the stellar spectrum at the planet's thermal wavelengths is on the Rayleigh–Jeans tail, faint and smooth, which is what makes secondary eclipses measurable.
- Wien's law also sets the *photochemistry*: an M dwarf with $\lambda_{\max} \simeq 0.8$–$1$ μm emits much less near-UV than the Sun, so photolysis of H_2O, CH_4, and O_3 proceeds differently on its planets, although flares and the chromospheric far-UV can compensate.

The Lambert surface: a surface of equal brightness

A **Lambert surface** scatters intensity *isotropically* (equally in all directions). It is the standard idealization of the reflectivity of planetary bodies, and it clarifies the difference between the intensity *emanating* from an object and the intensity *measured* by a distant observer.

- “Equal intensity in all directions” means the apparent surface brightness of an area element is the same from every viewing angle.
- Example: a sheet of white paper illuminated from above looks equally bright from any direction. A light bulb held to a white wall makes a bright spot whose location and extent do not change with viewing angle.
- Contrast with a mirror (specular reflection) or with ice crystals that strongly backscatter: the brightness then depends strongly on the viewing direction.

We now show that a plane Lambert surface, illuminated along its normal, has the same apparent brightness from any direction, and see *why* this is consistent with fewer photons leaving at slant angles.

Two observers of a Lambert surface

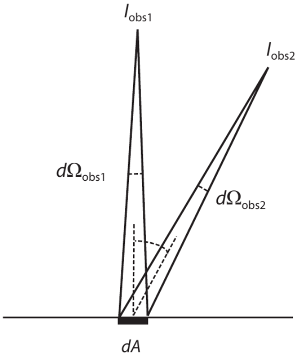

Figure 10: Intensity observed by a distant viewer. The solid angle subtended by dA is smaller at observer 2 than at observer 1. Reproduced from Seager (2010), Fig. 2.8.

Consider a plane Lambert surface illuminated along its normal. Observer 1 looks along the normal, observer 2 at an angle ϑ from it; both view the same differential area dA from a distance much larger than the size of the surface element. From the definition of intensity,

$$I_{\mathrm{obs1}} = \frac{\mathrm{d}E_{\mathrm{obs1}}}{\mathrm{d}A\,\mathrm{d}\Omega_{\mathrm{obs1}}\,\hat{\mathbf{n}}_{\mathrm{obs1}}\cdot\hat{\mathbf{k}}},$$

$$I_{\mathrm{obs2}} = \frac{\mathrm{d}E_{\mathrm{obs2}}}{\mathrm{d}A\,\mathrm{d}\Omega_{\mathrm{obs2}}\,\hat{\mathbf{n}}_{\mathrm{obs2}}\cdot\hat{\mathbf{k}}},$$

with, by the definition of the coordinate system, $\hat{\mathbf{n}}_{\mathrm{obs1}}\cdot\hat{\mathbf{k}} = 1$ and $\hat{\mathbf{n}}_{\mathrm{obs2}}\cdot\hat{\mathbf{k}} = \cos\vartheta$.

- **Geometry:** the solid angle subtended by dA at observer 2 is smaller than at observer 1 by the factor $\cos\vartheta$ (it approaches zero as $\vartheta \to 90°$, when the element is seen edge-on).

Why a Lambert surface looks equally bright from all angles

- **Physics (Lambert's cosine law):** for isotropic scattering the energy leaving per unit *projected* area is the same in every direction, so the energy (or number of photons) emerging from $\mathrm{d}A$ toward observer 2 drops as $\cos\vartheta$ away from the normal:

$$\mathrm{d}E_{\mathrm{obs2}} = \cos\vartheta \,\mathrm{d}E_{\mathrm{obs1}}.$$

Putting the two together, from observer 2

$$I_{\mathrm{obs2}} = \frac{\mathrm{d}E_{\mathrm{obs1}}\cos\vartheta}{\mathrm{d}A\,\mathrm{d}\Omega_{\mathrm{obs2}}\cos\vartheta} \qquad \Longrightarrow \qquad \boxed{I_{\mathrm{obs1}} = I_{\mathrm{obs2}}}$$

by comparison with the expression for observer 1.

- For a plane Lambert surface the intensity, i.e. the *brightness*, appears the same from all directions: the smaller number of photons emerging in the slant direction is exactly compensated by the smaller subtended angle of the area element.
- The *flux* received by observer 2 does decrease as $\cos\vartheta$; only the intensity is invariant. This is why a Lambert disk viewed obliquely looks just as bright but appears foreshortened.
- The same reasoning applied to a *sphere* gives the geometric albedo $2/3$ and the phase curve of a Lambert sphere (Chapter 3): each surface element is Lambertian, but the elements near the limb are lit obliquely, so they scatter fewer photons per unit area.

Chapter 2 summary and the SI radiometry vocabulary

What we defined.

- Intensity I (energy per area, solid angle, time, frequency); mean intensity J; flux $F = \oint I\,\hat{\mathbf{n}}\cdot\hat{\mathbf{k}}\,\mathrm{d}\Omega$; second moment K.
- Surface flux $F_\mathrm{S} = \pi I_\mathrm{S}$ for uniform intensity; observed flux $F_\oplus = (R_p/D_\oplus)^2 F_\mathrm{S}$.
- Luminosity $L = 4\pi R_p^2 F_\mathrm{S}$; hemispheric energy $E_\mathrm{S} = 2\pi R_p^2 F_\mathrm{S}$.
- Incident flux $F_\mathrm{inc} = (R_*/a)^2 F_{\mathrm{S},*}$; incident energy $E_\mathrm{inc} = \pi R_p^2 (R_*/a)^2 F_{\mathrm{S},*}$.
- Blackbody: $B(T,\nu)$, flux πB.
- Lambert surface: isotropic scattering, equal brightness from all directions.

SI radiometry names (Table 2.1) differ from the astronomical usage; the correspondence is:

SI quantity	unit	our name
radiant flux	W	luminosity, L
radiant intensity	$\mathrm{W\,sr^{-1}}$	—
radiance	$\mathrm{W\,sr^{-1}\,m^{-2}}$	intensity (total)
spectral radiance	$\mathrm{W\,sr^{-1}\,m^{-2}\,Hz^{-1}}$	intensity I_ν
irradiance	$\mathrm{W\,m^{-2}}$	incident flux
radiant exitance	$\mathrm{W\,m^{-2}}$	surface flux F_S
spectral irradiance	$\mathrm{W\,m^{-2}\,Hz^{-1}}$	flux F_ν

Atmospheric scientists say *radiance/irradiance* for intensity/flux; astronomers say intensity/flux. Same physics.

From flux to temperature: the plan for Chapter 3

When observing a distant exoplanet, the only thing we can measure is the radiation coming from it: a *flux*. The quantity relevant for the physics and chemistry of the atmosphere is its *temperature*.

- Only a single flux may be measured, yet real planets rarely have a single temperature. Which temperature should represent the planet? Which temperature should represent the flux?
- Temperature and albedo are tied together by **energy balance**: what is not reflected is absorbed and must be reradiated.
- Planet-to-star flux ratios (next lecture) follow from the temperatures and albedos, and decide what kind and size of telescope is needed to detect a given planet.

This section: energy balance $\rightarrow$ three planetary temperatures (T_{eff}, T_{eq}, T_{b}) $\rightarrow$ four albedos (single-scattering, geometric, spherical, Bond) and the relations among them.

Energy balance

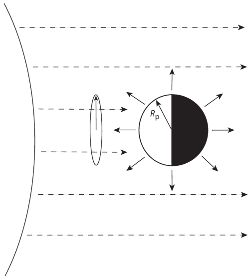

Figure 11: Planetary energy balance. Stellar radiation falls on the planet's cross-section πR_p^2; a fraction $(1 - A_B)$ is absorbed, here advected around the planet and reemitted uniformly into 4π. Reproduced from Seager (2010), Fig. 3.1.

No energy is created or destroyed in the atmosphere. All the energy comes either from the star (absorbed incident radiation) or from the planetary interior:

$$\boxed{E_{\text{out}}(t) = (1 - A_{\text{B}})\,E_{\text{inc}}(t) + E_{\text{int}}(t)}$$

- E_{out}: energy per unit time leaving the planet.
- E_{inc}: stellar energy per unit time incident on the planet.
- $(1 - A_{\text{B}})$: fraction absorbed, heating the atmosphere or surface, later reemitted at longer wavelengths.
- A_{B}: the **Bond albedo**, the fraction of incident stellar energy scattered back into space.
- E_{int}: energy per unit time supplied by the interior.

Energy balance in terms of fluxes

Write the outgoing energy as the planetary luminosity $L_p = 4\pi R_p^2 F_{\mathrm{S},p}$ and the incident energy with the result of Chapter 2, $E_{\mathrm{inc}} = \pi R_p^2 (R_*/a)^2 F_{\mathrm{S},*}$:

$$\boxed{4\pi R_p^2 F_{\mathrm{S},p} = (1 - A_{\mathrm{B}}) F_{\mathrm{S},*} \left(\frac{R_*}{a}\right)^2 \pi R_p^2 + L_{p,\mathrm{int}}}$$

Two energy sources for the atmosphere: absorbed (and reradiated) starlight, and the planet's own interior luminosity $L_{p,\mathrm{int}}$, the flux passing through a surface enclosing the planet from the inside.

- **Giant planets** (H/He): $L_{p,\mathrm{int}}$ is the gradual loss of residual gravitational potential energy from the planet's formation (Kelvin–Helmholtz contraction, plus helium rain in Saturn). Jupiter's internal luminosity is more than *twice* its reradiated absorbed sunlight — an indication of how long it takes energy to travel from a giant planet's interior out to space.
- **Earth:** $L_{p,\mathrm{int}}$ ($\simeq 44$ TW) comes partly from residual accretion energy but mostly from the decay of radioactive isotopes of uranium, thorium, and potassium. It is $\sim 2 \times 10^{-4}$ of the absorbed sunlight (1.2×10^{17} W, exercise 3.1b): negligible for the surface temperature, but it drives plate tectonics and volcanism, and thereby the long-term atmospheric evolution.

When can the interior be neglected?

For many planets the luminosity from external heating overwhelms the interior luminosity by many orders of magnitude:

- **Hot Jupiters** in very short-period orbits ($P < 4\,\mathrm{d}$): with $T_{\mathrm{eq}} = 1600\,\mathrm{K}$ and a Jupiter-like interior heat flow ($T_{\mathrm{eff,int}} \simeq 120\,\mathrm{K}$), the ratio of interior to absorbed stellar energy is $(120/1600)^4 \simeq 3 \times 10^{-5}$ (exercise 3.1a).
- **Terrestrial planets**, with their typically limited interior energy (Earth: 2×10^{-4}).
- **Exceptions:** young giant planets and brown dwarfs (directly imaged planets are seen *because* of their interior luminosity), and the Solar System giants, where the interior contributes 30–100% of the emitted power.

We therefore usually write the energy balance without the interior term,

$$\boxed{4\pi R_p^2 F_{\mathrm{S},p} = (1 - A_{\mathrm{B}})\, F_{\mathrm{S},*} \left(\frac{R_*}{a}\right)^2 \pi R_p^2}$$

Time dependence and eccentric orbits

- The planet exists in an equilibrium with the incident stellar radiation in which the heating is constant in time, so from now on we drop the time dependence of the planet flux and related quantities. (Energy per unit time is technically called *power*.)
- Planets on **eccentric orbits** do *not* have constant heating: the incident flux varies as $1/r^2$ between periastron and apastron, by a factor $[(1+e)/(1-e)]^2$, which is 2.3 for $e = 0.2$ and 1000 for HD 80606b ($e = 0.93$).
- Exercise 3.3: replacing a by the instantaneous distance r and averaging the incident flux over the orbit gives the orbit-averaged equilibrium temperature $\langle T_{\rm eq} \rangle = T_{\rm eq}(a)\,(1-e^2)^{-1/8}$, a weak correction because the planet spends most of its time near apastron.
- Whether the atmosphere follows the instantaneous heating or the average depends on the **radiative timescale** of the photosphere compared with the orbital period: hot, low-pressure photospheres respond within hours (HD 80606b's 8 μm flux rises by a factor of 5 within 6 hours of periastron passage), whereas deep, cold layers average over many orbits.

Three temperatures for a planet

There is no single temperature that describes a planet atmosphere: it varies with altitude, with horizontal position, and in time (day/night, seasons). Yet temperature governs, to first order, the chemical equilibrium state and hence the emergent spectrum. Three temperatures are in common use:

- **Effective temperature** T_{eff} — a proxy for the global temperature; defined from the *total* emitted flux. In principle a measured quantity.
- **Equilibrium temperature** T_{eq} — the T_{eff} a planet with no internal luminosity would have in equilibrium with its star. A *theoretical* number computed from R_*, $T_{\mathrm{eff},*}$, a, and A_{B}.
- **Brightness temperature** $T_{\mathrm{b}}(\nu)$ — the temperature of a blackbody with the same flux *at a given frequency*. The only temperature we can currently *measure* for exoplanets.

All three are defined by comparing the planet to a blackbody of the same size and distance; they differ in *which* flux is matched (total or monochromatic, measured or theoretical).

The effective temperature $T_{\rm eff}$

$T_{\rm eff}$ is the temperature of a blackbody of the same shape and at the same distance as the planet with the *same total flux* (integrated over all frequencies). Using the blackbody flux πB and assuming a uniform surface flux,

$$F_{\rm S} \equiv \int_0^\infty F_{\rm S}(\nu)\,{\rm d}\nu = \pi \int_0^\infty B(T,\nu)\,{\rm d}\nu \equiv \sigma_{\rm R} T_{\rm eff}^4.$$

Carrying out the integral (exercise 3.2, using $\int_0^\infty u^3/({\rm e}^u - 1)\,{\rm d}u = \pi^4/15$) gives the **Stefan–Boltzmann law** and constant

$$\boxed{F_{\rm S} = \sigma_{\rm R} T_{\rm eff}^4, \qquad \sigma_{\rm R} = \frac{2\pi^5 k_{\rm B}^4}{15\,c^2 h^3} = 5.670\times10^{-8}\ {\rm J\,K^{-4}\,m^{-2}\,s^{-1}}}$$

- $\sigma_{\rm R}$ is not a fundamental constant: it is built from h, c, and $k_{\rm B}$. (Heng, problem 2.8.4: the same $\sigma_{\rm R} T^4$ results whether one integrates B_λ, B_ν, or $B_{\tilde\nu}$.)
- In principle $T_{\rm eff}$ is a *measured* quantity: take the total flux of the planet, convert it to surface flux with $(D_\oplus/R_p)^2$, and find the temperature of the blackbody with the same total flux. In practice we never observe *all* wavelengths, so a model is needed to fill the gaps.
- $F_{\rm S}$ here is the planetary surface flux, in ${\rm J\,m^{-2}\,s^{-1}}$, radiated at the planet, as derived in Chapter 2.

What T_{eff} means physically

- T_{eff} describes the global temperature of the planet at the altitude where the *bulk of the radiation leaves* the planet or star. This altitude is sometimes called the planet's **photosphere** or "surface", whether or not it is at the solid surface.
- T_{eff} may be very different from the surface temperature. **Venus:** $T_{\mathrm{eff}} \simeq 230\,\mathrm{K}$, a value that comes from the cloud tops which obscure the solid surface. Because of a strong greenhouse effect the surface temperature is $730\,\mathrm{K}$, almost $500\,\mathrm{K}$ hotter than T_{eff}. Earth: $T_{\mathrm{eff}} \simeq 255\,\mathrm{K}$ versus a mean surface temperature of $288\,\mathrm{K}$, a greenhouse warming of $33\,\mathrm{K}$.
- T_{eff} varies widely among planets. The coldest planets of the Solar System, Uranus and Neptune, have $T_{\mathrm{eff}} = 59\,\mathrm{K}$; Jupiter $124\,\mathrm{K}$; Earth $\simeq 255\,\mathrm{K}$. If we could measure T_{eff} for exoplanets, we would expect the hottest hot Jupiters to have $T_{\mathrm{eff}} > 2000\,\mathrm{K}$, and ultra-hot Jupiters such as KELT-9b exceed $4000\,\mathrm{K}$ on the dayside.
- Because T_{eff} is defined by the *total* flux, it says nothing about the spectral shape (next slide).

$T_{\rm eff}$ applied to two stars

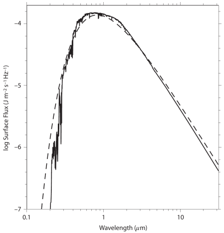

Figure 12: Surface flux of a $T_{\rm eff} = 6130$ K G0IV star (HD 149026; solid) and the 6130 K blackbody (dashed). Seager (2010), Fig. 3.2.

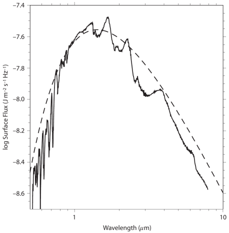

Figure 13: Same for a $T_{\rm eff} = 3600$ K M2.5V star (GJ 436); the deep features are TiO (visible) and H_2O (near-IR). Seager (2010), Fig. 3.3.

Both stars have the same total flux as their blackbodies (equal areas under the curves). For the G star the spectrum is close to a blackbody; for the M star it departs strongly. *Lesson:* $T_{\rm eff}$ says nothing about the spectral shape. A planet around GJ 436 receives far more near-infrared and far less blue light than a blackbody would suggest, which matters for the Bond albedo (weighted by the stellar spectrum), for the photochemistry (little UV), and for the interpretation of secondary-eclipse depths (the stellar brightness temperature in the bands is not $T_{\rm eff}$).

The equilibrium temperature T_{eq}

T_{eq} is an estimate of T_{eff} for a planet with *no internal luminosity*: the effective temperature attained by an isothermal planet in complete equilibrium with the radiation of its star. It refers to no flux measurement and is essential for exoplanets, for which T_{eff} is hard to measure. Start from energy balance, with a correction factor f to the 4π so that we can describe one hemisphere:

$$\frac{4\pi}{f} R_p^2 F_{\mathrm{S},p} = (1 - A_{\mathrm{B}}) F_{\mathrm{S},*} \left(\frac{R_*}{a}\right)^2 \pi R_p^2.$$

Insert the Stefan–Boltzmann law $F_{\mathrm{S},p} = \sigma_{\mathrm{R}} T_{\text{eq}}^4$ and $F_{\mathrm{S},*} = \sigma_{\mathrm{R}} {T_{\text{eff},*}}^4$ and define $f' = f/4$:

$$\boxed{T_{\text{eq}} = T_{\text{eff},*} \left(\frac{R_*}{a}\right)^{1/2} \left[f'(1 - A_{\mathrm{B}})\right]^{1/4}}$$

- **Scaling:** $T_{\text{eq}} \propto a^{-1/2}$ and $T_{\text{eq}} \propto R_*^{1/2} T_{\text{eff},*}$; the albedo enters only as $(1 - A_{\mathrm{B}})^{1/4}$ (a factor 0.9 for $A_{\mathrm{B}} = 0.3$).
- Since $L_* \propto R_*^2 {T_{\text{eff},*}}^4$, equivalently $T_{\text{eq}}^4 = f'(1 - A_{\mathrm{B}}) L_* / (4\pi\sigma_{\mathrm{R}} a^2)$: the temperature is set by the stellar flux at the planet's orbit.

The redistribution factor f'

For exoplanets we observe only one hemisphere at a time; this is what drives the need for f'. It describes *how the absorbed stellar energy is distributed* over the planet before being reradiated.

- **Uniform redistribution** ($f' = 1/4, f = 1$): the absorbed energy is spread over the whole sphere ($4\pi R_p^2$) and T_{eq} is the same for any hemispheric view. For Earth: $T_{\text{eq}} = 5778\,\text{K}\,(R_\odot/\text{AU})^{1/2}[0.7/4]^{1/4} = 255\,\text{K}$.
- **Instantaneous reradiation, no advection** ($f' = 2/3, f = 8/3$): a slowly rotating or tidally locked planet absorbs energy only on the dayside and each element reradiates locally. The dayside hemisphere is then much hotter: $(8/3)^{1/4} = 1.28$ times the uniform value. The value $2/3$ is the same number as the geometric albedo of a Lambert sphere, and arises from the same $\cos\vartheta$-weighted hemispheric integral: the observer sees the hot substellar region face-on and the cooler limb obliquely.
- Between these two end cases there is no simple analytic form. To estimate the actual dayside temperature one needs observations (phase curves) or atmospheric circulation models (Chapter 10).
- **Caution:** f has many different definitions in the literature ($f = 1/4$, $1/2$, $2/3$, $\ldots$ meaning different things). Seager's choice $f' = f/4$ as a correction to 4π is deliberate, for symmetry with the albedo derivation for scattered light from the dayside.

T_{eq} versus orbital distance

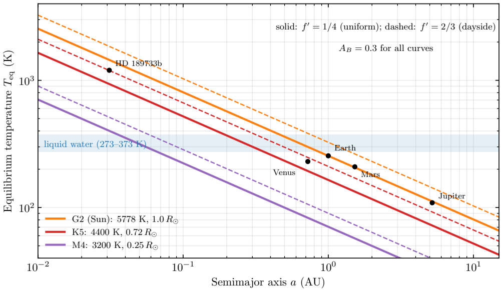

Figure 14: Equilibrium temperature versus semimajor axis for three host stars, with $A_B = 0.3$; solid $f' = 1/4$, dashed $f' = 2/3$. Points are Solar System T_{eq} values (Table 3.1) and HD 189733b.

- The $a^{-1/2}$ slope: moving a planet 100 times closer raises T_{eq} by a factor of 10 (Earth at 0.01 AU would be ~ 2500 K).
- Around an M4 dwarf the same T_{eq} is reached at $a \simeq 0.05$ AU: the habitable zone shrinks inward together with T_{eq}, which is why M-dwarf habitable-zone planets transit frequently and are tidally locked.
- HD 189733b ($a = 0.031$ AU, K dwarf): $T_{eq} \simeq 1200$ K for $f' = 1/4$, close to the measured dayside brightness temperature (Lecture 1), indicating efficient day–night heat transport.
- The dashed curves show how much hotter a non-redistributing dayside can be.

How good is T_{eq}? The Solar System test

Planet	T_{eff} (K)	T_{eq} (K)	A_g (500 nm)	A_B
Mercury	—	—	0.106	0.119
Venus	~ 230	230	0.65	0.750
Earth	~ 255	253	0.367	0.306
Mars	~ 212	209	0.150	0.250
Jupiter	124.4 ± 0.3	109	0.52	0.343
Saturn	95.0 ± 0.4	80	0.47	0.342
Uranus	59.1 ± 0.3	58	0.51	0.300
Neptune	59.3 ± 0.8	46	0.42	0.290

Seager (2010), Table 3.1; values from Cox (2000). T_{eq} uses the measured A_B, the known a, and $f' = 1/4$.

- For the terrestrial planets T_{eff} and T_{eq} agree to within a few kelvin: their interior heat is negligible.
- For Jupiter, Saturn, and Neptune $T_{eff} > T_{eq}$ by 15–30%, the signature of their **internal luminosity** ($T_{eff}^4 = T_{eq}^4 + T_{int}^4$ with $T_{int} \simeq 100\,\mathrm{K}$ for Jupiter). Uranus is the exception: $T_{eff} \simeq T_{eq}$, so its interior heat flow is anomalously small — still not fully understood.
- Note that $A_g \neq A_B$ in general, and that A_g can exceed A_B (Jupiter, Uranus, Neptune): the two albedos measure different things (next section).

The brightness temperature $T_b(\nu)$: motivation

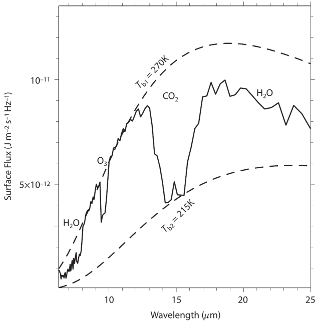

Figure 15: Brightness temperature. Solid: Earth's surface flux measured by Mars Global Surveyor. Dashed: blackbody fluxes at two temperatures. Reproduced from Seager (2010), Fig. 3.4.

The flux of exoplanets at different wavelengths or frequencies is the quantity we can measure at Earth, $F_\oplus = (R_p/D_\oplus)^2 F_S$ for a very distant planet with surface flux F_S.

- Our intuitive familiarity with temperature makes it easier to use a temperature instead of a flux to describe exoplanet atmosphere measurements.
- Temperatures are more directly relevant than fluxes for the physics and chemistry of the atmosphere.
- Fluxes depend on the planet's distance from Earth and so change from one planet to the next, whereas the temperature of a planet atmosphere is distance independent.

Measured exoplanet fluxes are therefore often stated as **brightness temperatures**: the temperature of a blackbody of the same shape and at the same distance as the planet with the *same flux in a specified frequency range*.

The brightness temperature $T_{\mathrm{b}}(\nu)$: definition

$$\boxed{\pi B\big(T_{\mathrm{b}}(\nu),\nu\big) = F_{\mathrm{S}}(\nu) = F_{\oplus}(\nu)\left(\frac{D_{\oplus}}{R_p}\right)^2}$$

- $T_{\mathrm{b}}(\nu)$ is a frequency-dependent expression of the planetary flux: invert the Planck function at the observed frequency for the temperature. Seager writes this with the Stefan–Boltzmann form $\sigma_{\mathrm{R}} T_{\mathrm{b}}^4(\nu) = F_{\mathrm{S}}(\nu)$ as shorthand for "convert flux to temperature"; the monochromatic Planck inversion is the operational definition used in practice.
- The brightness temperature assumes that the planet behaves as a blackbody *only* in the frequency range where the flux is measured; nothing is assumed about other frequencies.
- While a blackbody radiator has a constant $T_{\mathrm{b}}(\nu)$, planets with spectra that depart from a blackbody have a $T_{\mathrm{b}}(\nu)$ that can vary widely with wavelength or frequency.
- In the Rayleigh–Jeans regime T_{b} is simply proportional to the flux, $T_{\mathrm{b}} = c^2 F_{\mathrm{S}}(\nu)/(2\pi\nu^2 k_{\mathrm{B}})$, which is the definition used in radio astronomy; in the Wien regime a small change in flux corresponds to a small change in T_{b}, so brightness temperatures are a compressed, robust way to quote infrared fluxes.

Reading brightness temperatures

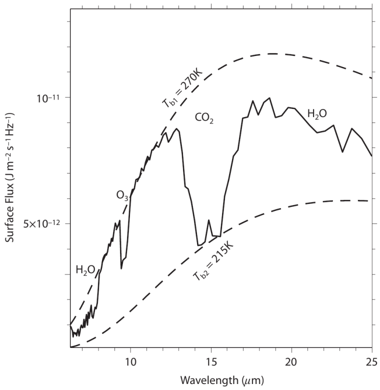

Figure 16: Earth's mid-infrared spectrum: $T_\mathrm{b}(12\,\mu\mathrm{m}) \simeq 270\,\mathrm{K}$ but $T_\mathrm{b}(14.2\,\mu\mathrm{m}) \simeq 215\,\mathrm{K}$. Reproduced from Seager (2010), Fig. 3.4.

- For Earth: $T_\mathrm{b} \simeq 270\,\mathrm{K}$ in the $12\,\mu\mathrm{m}$ window, where we see the warm surface through a transparent atmosphere, but $T_\mathrm{b} \simeq 215\,\mathrm{K}$ in the 14–15 μm CO_2 band, where the atmosphere is opaque and we see the cold tropopause region. The 9.6 μm O_3 band gives an intermediate value from the stratosphere.
- $T_\mathrm{b}(\nu)$ is the *only* temperature we can currently measure for exoplanets, because it requires flux at just one frequency or wavelength range: a secondary-eclipse depth in the Spitzer 3.6, 4.5, 5.8, and 8 μm bands gives four brightness temperatures of the dayside.
- To get from a measured $T_\mathrm{b}(\nu)$ to the more global T_eff we must resort to a model atmosphere code that fills in the unobserved wavelengths.

Relating the three temperatures

	definition	measured or theoretical?	typical use
$T_{\rm eff}$	blackbody with the same *total* flux: $\sigma_{\rm R} T_{\rm eff}^4 = F_{\rm S}$	measurable in principle (needs all λ)	global energy output; interior heat
$T_{\rm eq}$	$T_{\rm eff}$ of a planet with no interior heat, from R_*, $T_{{\rm eff},*}$, a, $A_{\rm B}$, f'	theoretical	first estimate for any new planet
$T_{\rm b}(\nu)$	blackbody with the same flux *at* ν	measured (one band suffices)	reporting eclipse depths; vertical structure via $T_{\rm b}(\nu)$

- $T_{\rm eff}^4 = T_{\rm eq}^4 + T_{\rm int}^4$ when interior heat matters (the two fluxes add); $T_{\rm eff} \simeq T_{\rm eq}$ otherwise, provided the right f' is used for the hemisphere observed.
- All three compare the planet with a blackbody of the same size and distance; they differ only in *which* flux is matched: total or monochromatic, measured or computed.

Temperatures of a tidally locked planet

- $T_{\mathrm{b}}(\nu)$ scatters around T_{eff}: it is higher in spectral windows that see deep, warm layers and lower in strong absorption bands that see high, cold layers, for a temperature decreasing with altitude. A temperature *inversion* reverses this: the bands appear in *emission* and T_{b} is higher inside them. Thus $T_{\mathrm{b}}(\nu)$ across a spectrum is a crude thermometer of the vertical structure, which Chapter 9 turns into a quantitative retrieval.
- For a tidally locked hot Jupiter the *dayside* T_{b} measured at secondary eclipse can exceed $T_{\mathrm{eq}}(f' = 1/4)$ by up to $(8/3)^{1/4} = 1.28$ if the absorbed energy is not redistributed, whereas the *nightside* T_{b} measured near transit can be far below it. The day–night contrast, i.e. the effective f', is a diagnostic of the winds (Chapter 10).
- Example: HD 189733b has $T_{\mathrm{eq}}(f' = 1/4) \simeq 1200\,\mathrm{K}$; the measured $8\,\mu\mathrm{m}$ brightness temperatures are $\simeq 1210\,\mathrm{K}$ on the dayside and $\simeq 970\,\mathrm{K}$ on the nightside (Knutson et al. 2007), so heat redistribution is efficient. For WASP-18b, in contrast, the dayside is close to the $f' = 2/3$ limit and the nightside is very cold.
- A single measured T_{b} therefore does not determine A_{B}: with f' unknown, albedo and redistribution are degenerate in the dayside temperature. Breaking the degeneracy needs a phase curve or an optical eclipse depth (Lecture 3).

What makes a planet bright or dark?

The **albedo** is the ratio of the light scattered by a planet to the light received by it. As for temperature, there is no single albedo; but the albedo is an important parameter because the reflectivity is indicative of cloud or surface conditions, and because it controls the energy balance and hence T_{eff}.

- **Venus** is the brightest planet, scattering 0.75 of the incident energy: 100% coverage by H_2SO_4 clouds.
- **Earth** has highly reflective water/ice clouds but only $\sim 50\%$ cover, and scatters 0.3.
- **Icy bodies** are bright, fresh ice more than old ice. Enceladus, with water geysers producing fresh ice, is extremely bright.
- **Mercury**, with no atmosphere, clouds, or ice, scatters only 0.1. Hot Jupiters are similarly dark ($A_g \lesssim 0.1$): no bright cloud decks, and strong Na/K absorption.

Four albedos and their relations: **single-scattering** $\tilde{\omega}$ (one particle), **geometric** $A_g(\nu)$ (full phase, relative to a Lambert disk), **spherical** $A_\mathrm{S}(\nu)$ (all directions, one frequency), **Bond** A_B (all directions, all frequencies) — plus the **apparent** albedo, the one that can be measured for exoplanets.

The single-scattering albedo $\tilde{\omega}$

- The single-scattering albedo $\tilde{\omega}$ is the fraction of incident light that is scattered (rather than absorbed) by a *single particle* in the atmosphere or on the surface. It is wavelength dependent.
- Examples: a water-ice crystal in Earth's atmosphere can reach $\tilde{\omega} \simeq 0.8$; a plant leaf at visible wavelengths is closer to 0.1. Old to fresh snow 0.45–0.85; clouds 0.35–0.8; desert sand 0.2–0.35; ocean water < 0.05.
- $\tilde{\omega}$ is *not* a global albedo of the planet. Nevertheless, the reflective properties of the individual gas, cloud, and surface particles ultimately produce the planet's global albedo through complex *multiple* scattering of the incident radiation.
- Equally important is the *directional* scattering property of a particle. Few particles in nature, if any, scatter isotropically; ice crystals, for example, have a strong tendency to backscatter. For a surface this is described by the bidirectional reflectance distribution function (BRDF); for a single particle by the phase function $P(\Theta)$ (Chapter 5).
- In radiative-transfer language (Lecture 3), $\tilde{\omega} = \sigma_s/(\alpha + \sigma_s)$, the ratio of the scattering coefficient to the total extinction. Conservative scattering, $\tilde{\omega} = 1$, is the Lambert-sphere idealization; a Rayleigh-scattering atmosphere over a black surface has $\tilde{\omega} \to 1$ in the blue and $\to 0$ in the red.

The phase angle α

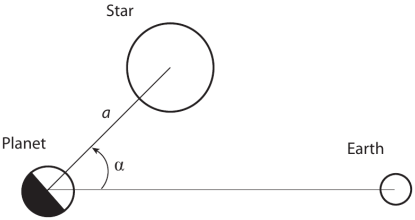

Figure 17: The phase angle α is the star–planet–observer angle; a is the semimajor axis. Reproduced from Seager (2010), Fig. 3.5.

Before defining the planetary albedos we need the **phase angle**: the star–planet–observer angle.

- $\alpha = 0^\circ$ corresponds to full phase: the observer sees the entire illuminated hemisphere.
- $\alpha = 90^\circ$ is quadrature (“half” phase); this is where the planet’s angular separation from the star is largest for a circular orbit, the natural configuration for direct imaging.
- $\alpha > 170^\circ$ corresponds to a thin crescent, and at $\alpha = 180^\circ$ the planet is not illuminated at all as seen by the observer (it is between the observer and the star: transit).

Phase angles in the Solar System and for exoplanets

- **Solar System:** only Mercury and Venus, interior to Earth's orbit, show all illumination phases from Earth; the outer giant planets show only a few to several degrees of phase angle. Full phase, $\alpha \simeq 0$, is the natural configuration for observing Solar System planets, with the Sun, Earth, and planet aligned at opposition.
- For exoplanets the range of visible planetary phases depends on the **orbital inclination** of the planet–star system.
- A planet in a system with $i = 90^\circ$ (edge-on) shows all phases as seen from Earth, except phases right near $\alpha = 0$ when the planet is directly behind the star (secondary eclipse) and near $\alpha = 180^\circ$ when it is in front (transit).
- In contrast, a planet with a face-on orbit ($i = 0^\circ$) has a constant "half" phase, $\alpha = 90^\circ$, throughout its orbit. In general the accessible range is $90^\circ - i \leq \alpha \leq 90^\circ + i$.

Full phase is never quite observable

- Because of the randomness of exoplanet orbital inclinations, $\alpha \approx 0$ may never occur for some planets.
- In reality $\alpha \equiv 0$ does not occur for any exoplanet, because the planet would be directly behind its star and not observable; however, we consider a small α close enough to $\alpha = 0$ for our purposes — for a transiting planet, the moments just before and after secondary eclipse, where $\alpha \simeq R_*/a \sim 5^\circ$.
- The phase angle is related to the orbital phase: for a circular edge-on orbit, $\cos\alpha = -\cos(2\pi\varphi)$ with φ the orbital phase measured from transit, so that α sweeps from 180° at transit through 90° at quadrature to 0 at eclipse. For inclination i, $\cos\alpha = -\sin i \cos(2\pi\varphi)$.
- For directly imaged planets the situation is reversed: small α (planet behind or in front of the star) is inaccessible because of the angular proximity to the star, and the accessible phases cluster around quadrature. Reflected-light imaging therefore measures the apparent albedo near $\alpha \simeq 90^\circ$ rather than A_g.

The geometric albedo $A_g(\nu)$: definition

The geometric albedo is the ratio of a planet's flux at *zero phase angle* (full phase) to the flux from a **Lambert disk** at the same distance and with the same cross-sectional area (i.e. subtending the same solid angle).

- The definition is historical: the brightness of a Solar System planet on a photograph was compared to that of a uniformly illuminated Lambert disk, a *relative* measurement that cancels instrumental errors.
- $A_g(\nu)$ is *not* the fraction of incident energy scattered at $\alpha = 0$ (exercise 3.5: that fraction is A_g/π).

Numerator: the planet flux at Earth at $\alpha = 0$, from Chapter 2 in latitude–longitude coordinates, keeping only *scattered* intensity (not thermal reemission):

$$F_{\oplus}(\nu) = \left(\frac{R_p}{D_{\oplus}}\right)^2 \int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} I_{\mathrm{S,scat}}(\theta,\phi,\nu)\cos\phi\cos^2\theta \,\mathrm{d}\theta\,\mathrm{d}\phi.$$

Denominator: a Lambert disk of radius R_p scatters isotropically with $I_{\mathrm{S,scat}} = I_{\mathrm{inc}}$, uniform over the disk, so

$$F_{\mathrm{L\,disk}}(\nu) = \left(\frac{R_p}{D_{\oplus}}\right)^2 \pi I_{\mathrm{inc}}(\nu).$$

The geometric albedo: general expression

Dividing numerator by denominator,

$$\boxed{A_g(\nu) = \frac{\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} I_{\mathrm{S,scat}}(\theta,\phi,\nu)\cos\phi\cos^2\theta\,\mathrm{d}\theta\,\mathrm{d}\phi}{\pi I_{\mathrm{inc}}(\nu)}}$$

- Something interesting: the denominator $\pi I_{\mathrm{inc}}(\nu) = F_{\mathrm{inc}}(\nu)$ is the incident stellar flux at the *substellar point*. So A_g may also be described as the ratio of the planet's scattered flux to the incident stellar flux at the substellar point.
- This is the form to use to compute A_g from the output of a *model atmosphere code*, which delivers the scattered intensity at the planetary surface as a function of location, $I_{\mathrm{S,scat}}(\theta,\phi,\nu)$.
- The subscript “scat” means that only *scattered* radiation is counted, not radiation that has been absorbed and thermally reradiated; at visible wavelengths, for planets cooler than $\sim 1500\,\mathrm{K}$, the distinction is clean.

The geometric albedo: making the physics explicit

For a more conceptual understanding, relate the scattered intensity to the incoming intensity with the dimensionless $p(\Theta)$: the fraction of incident intensity scattered out of the planet into the angle Θ from the incident direction. All the physics of directional and multiple scattering inside the atmosphere is buried in p (it is *not* the single-scattering phase function P of Chapter 5):

$$I_{\mathrm{S,scat}}(\theta,\phi,\nu) = I_{\mathrm{inc}}(\theta',\phi',\nu)\,p(\Theta),$$

primes for the direction of incidence, no primes after scattering. For uniform incident stellar intensity, diluted away from the substellar point by the projection (Fig. 2.3),

$$I_{\mathrm{inc}}(\theta',\phi',\nu) = \hat{\mathbf{n}}\cdot\hat{\mathbf{k}}\,I_{\mathrm{inc}}(\nu) = \cos\Theta\, I_{\mathrm{inc}}(\nu) = \cos\theta\cos\phi\, I_{\mathrm{inc}}(\nu),$$

so that $I_{\mathrm{S,scat}} = \cos\theta\cos\phi\, I_{\mathrm{inc}}(\nu)\,p(\theta,\phi,\theta',\phi')$. Then $I_{\mathrm{inc}}(\nu)$ cancels between numerator and denominator:

$$\boxed{A_g(\nu) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} p(\theta,\phi,\theta',\phi')\cos^2\phi\cos^3\theta\,\mathrm{d}\theta\,\mathrm{d}\phi}$$

One factor $\cos\phi\cos\theta$ comes from the oblique illumination, the other $\cos\phi\cos^2\theta$ from the flux projection and the solid angle toward the observer.

A_g of a Lambert sphere, and why A_g can exceed 1

For a **Lambert sphere** the intensity is scattered equally in all directions, $p(\Theta) = 1$, and

$$A_g^{\text{Lambert}} = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} \cos^2\phi \cos^3\theta \, \mathrm{d}\theta \, \mathrm{d}\phi = \frac{1}{\pi}\cdot\frac{\pi}{2}\cdot\frac{4}{3} = \boxed{\frac{2}{3}}$$

- $A_g = 2/3 < 1$ even though a Lambert sphere scatters *all* incident radiation and absorbs none. One third of the light is scattered out of the line of sight: the limb regions are illuminated obliquely *and* seen obliquely, so they contribute less than the corresponding parts of a flat disk.
- By the same definition, A_g *can be greater than 1*: it compares the planet not to the incident radiation but to a perfectly diffusing disk. **Enceladus** has $A_g = 1.38$ at visible wavelengths (Verbiscer et al. 2007): fresh ice backscatters strongly, so at $\alpha = 0$ it returns more light toward the observer than a Lambert disk would. Rayleigh-scattering atmospheres also backscatter more than Lambert ($A_g = 3/4$ for a semi-infinite conservative Rayleigh atmosphere).
- Measurements of A_g have been elusive for exoplanets (upper limits from MOST, CoRoT, Kepler; a few detections such as Kepler-7b with $A_g \simeq 0.3$), but A_g remains the standard way to classify theoretical models.

The Bond albedo A_B and the spherical albedo $A_\mathrm{S}(\nu)$

- A_B is the fraction of incident stellar *energy* scattered back into space, including radiation into *all* directions and at *all* frequencies. By definition $A_\mathrm{B} \leq 1$.
- The **spherical albedo** $A_\mathrm{S}(\nu)$ is the same quantity at a specific frequency. The Bond albedo is the spherical albedo averaged over frequency *weighted by the incident stellar spectrum*:

$$\boxed{A_\mathrm{B} = \frac{\int_0^\infty A_\mathrm{S}(\nu)\, F_{\mathrm{S},*}(\nu)\, \mathrm{d}\nu}{\int_0^\infty F_{\mathrm{S},*}(\nu)\, \mathrm{d}\nu}}$$

  (Seager writes $A_\mathrm{B} = \int A_\mathrm{S}\, \mathrm{d}\nu$ as shorthand and states that the weighting by the incident radiation is implied; Heng's Eq. 2.10 gives the weighted form explicitly.)
- Consequently both A_B and A_S depend on the *host star's spectrum*: the same planet has a different Bond albedo around a G star and around an M star, because an M star emits most of its light in the near-IR where the atmosphere may be dark.

To derive $A_\mathrm{S}(\nu)$ we must account for the light scattered at *every* phase angle, not just at $\alpha = 0$. At each α only a portion of the hemisphere facing the observer is illuminated (next slide).

Scattered flux at phase angle α

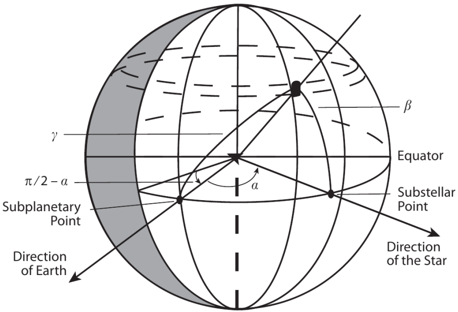

Figure 18: Coordinates for the phase-dependent scattered flux. θ is latitude from the intensity equator, ϕ longitude from the sub-Earth point, α the phase angle. After Sobolev (1975). Reproduced from Seager (2010), Fig. 3.6.

Measure longitude ϕ from the sub-Earth (subplanetary) point. The substellar point is then at longitude α, and a surface element at (θ, ϕ) is illuminated with the projection $\cos(\alpha-\phi)\cos\theta$ and seen with the projection $\cos\phi\cos\theta$. Only the longitudes $\alpha-\pi/2<\phi<\pi/2$ are both lit and visible. The scattered flux toward the observer is

$$F_{S,\alpha}(\nu)=\int_{\alpha-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} I_{\rm inc}\, p\,\cos(\alpha-\phi)\cos\phi\cos^3\theta\,d\theta\,d\phi,$$

and for uniform incident intensity we write

$$F_{S,\alpha}(\nu)=I_{\rm inc}(\nu)\,\Psi_\alpha(\nu),$$

where Ψ_α is the fraction of scattered radiation in the direction of the observer as a function of α. At $\alpha=0$ it reduces to the geometric-albedo integral: $\Psi_{\alpha=0}=A_g/\pi$.

The spherical albedo: integrating over all directions

To count the energy scattered into all directions, surround the planet with a sphere and use α as the polar angle and β as the azimuth (azimuthal symmetry assumed). The total scattered energy per unit time per frequency is

$$E_{\mathrm{scat}}(\nu) = I_{\mathrm{inc}}(\nu) \int_0^{2\pi} \int_0^{\pi} \Psi_\alpha(\nu)\, R_p^2 \sin\alpha \,\mathrm{d}\alpha\,\mathrm{d}\beta = 2\pi R_p^2 I_{\mathrm{inc}}(\nu) \int_0^{\pi} \Psi_\alpha(\nu) \sin\alpha\,\mathrm{d}\alpha.$$

The incident energy (Chapter 2) is $E_{\mathrm{inc}}(\nu) = \pi R_p^2 \cdot \pi I_{\mathrm{inc}}(\nu)$. Their ratio is the spherical albedo:

$$\boxed{A_{\mathrm{S}}(\nu) = \frac{E_{\mathrm{scat}}(\nu)}{E_{\mathrm{inc}}(\nu)} = \frac{2}{\pi} \int_0^{\pi} \Psi_\alpha(\nu) \sin\alpha\,\mathrm{d}\alpha}$$

- A_{S} needs Ψ_α at *all* phase angles. From Earth we cannot see all phases of most planets, so A_{S} (and A_{B}) are not directly measurable: we need a way to estimate them from the one thing we can hope to measure near full phase, A_g.
- Note the different weighting from T_{eq}: the factor $2/\pi$ is $2\pi R_p^2/(\pi R_p^2 \cdot \pi)$, the sphere over the disk and the π of the flux.

Relating A_S to A_g: the phase integral q

Using $\Psi_{\alpha=0}(\nu) = A_g(\nu)/\pi$, rewrite the spherical albedo as

$$A_S(\nu) = \frac{\pi A_g(\nu)}{\Psi_{\alpha=0}(\nu)} \cdot \frac{2}{\pi}\int_0^{\pi} \Psi_\alpha(\nu)\sin\alpha\,\mathrm{d}\alpha \quad \Longrightarrow \quad \boxed{A_S(\nu) = A_g(\nu)\,q(\nu)}$$

with the **phase integral**

$$q(\nu) \equiv \frac{A_S(\nu)}{A_g(\nu)} = 2\int_0^{\pi} \Phi_\alpha(\nu)\sin\alpha\,\mathrm{d}\alpha, \qquad \Phi_\alpha(\nu) \equiv \frac{\Psi_\alpha(\nu)}{\Psi_{\alpha=0}(\nu)}$$

where Φ_α is the **phase function**: the scattered flux versus phase angle normalized to full phase (this is the phase curve of the next lecture).

- A_S can be *estimated* from a measured A_g and a theoretical q. Heng writes the same thing as $A_S = A_g P_g$.
- **Lambert sphere:** $\Phi_\alpha = \frac{1}{\pi}[\sin\alpha + (\pi-\alpha)\cos\alpha]$ (derived next lecture), so $q = 3/2$ and

  $$A_g = \tfrac{2}{3}A_S, \qquad A_S = A_B = 1,\ A_g = \tfrac{2}{3}.$$

  All light is scattered back to space, so the spherical albedo is 1; the geometric albedo counts only the light scattered toward the observer at $\alpha = 0$ and is less.
- **Rayleigh scattering:** $q = 4/3$; a Lambert *plane* (exercise 3.6): $\Phi_\alpha = \cos\alpha$. Real planets: $q \simeq 1.2$–1.7.

Bond versus geometric albedo for the giant planets

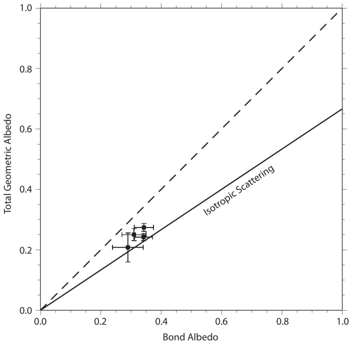

Figure 19: Total (wavelength-integrated) geometric albedo versus Bond albedo for a Lambert sphere (solid, $A_g/A_\mathrm{B} = 2/3$) and for Uranus, Neptune, Saturn, Jupiter (points, in order of increasing A_B). The dashed line is $A_g = A_\mathrm{B}$. Adapted from Rowe et al. (2006); reproduced from Seager (2010), Fig. 3.7.

- All four giants lie *above* the Lambert (isotropic scattering) line: their atmospheres backscatter more than a Lambert sphere, so $A_g/A_\mathrm{B} > 2/3$ ($q < 3/2$).
- The wedge between the Lambert line and the line $A_g = A_\mathrm{B}$ bounds the photometric properties of most spherical bodies with deep atmospheres (Rayleigh scattering, clouds, dust). A planet with deep atmosphere must lie to the right of the dashed line.
- Practical use: a measured (or upper-limit) A_g of an exoplanet, e.g. the MOST limit $A_g < 0.25$ for HD 209458b, can be converted to a range of A_B, hence to a range of T_eq.

The apparent albedo: the one an exoplanet can give us

The geometric, spherical, and Bond albedos are unsatisfactory for exoplanets because *none can be measured*:

- A_g is defined at full phase ($\alpha = 0$), when the planet is behind the star and unobservable (for transiting planets the eclipse depth measured just before/after eclipse is the practical substitute, at a small α).
- A_S and A_B require all phase angles. With direct imaging, phases where the planet projects too close to the star are inaccessible, and unless the orbit is edge-on the planet does not go through all phases at all.

Apparent albedo. The ratio of the scattered flux emerging from the planet toward the observer to the scattered flux from a *perfectly reflecting Lambert sphere* of the same size, at the same phase angle, at the same distance:

$$A_\mathrm{app}(\alpha,\nu) = \frac{F_{\mathrm{S},\alpha}(\nu)}{F^{\mathrm{Lambert}}_{\mathrm{S},\alpha}(\nu)} = \frac{A_g(\nu)\,\Phi_\alpha(\nu)}{\frac{2}{3}\,\Phi^{\mathrm{Lambert}}_\alpha} \qquad \text{(exercise 3.9).}$$

- It is the planet's reflectance in the direction of the observer at a given time, a function of phase angle and frequency.
- It is the only albedo that can be derived from a *single* observation of a planet of known size — a directly imaged planet at known separation and phase.

Take-home messages

Radiation quantities. Intensity I is the fundamental variable (per area, solid angle, time, frequency; constant along a ray in vacuum). Its moments are the mean intensity J, the flux $F = \oint I\,\hat{\mathbf{n}}\cdot\hat{\mathbf{k}}\,\mathrm{d}\Omega$, and K (radiation pressure). For uniform intensity the surface flux is $F_\mathrm{S} = \pi I_\mathrm{S}$; at Earth $F_\oplus = (R_p/D_\oplus)^2 F_\mathrm{S}$.

Energy budget. $L = 4\pi R_p^2 F_\mathrm{S}$; incident flux $(R_*/a)^2 F_{\mathrm{S},*}$ at the substellar point; incident energy $\pi R_p^2 (R_*/a)^2 F_{\mathrm{S},*}$ (cross-section, not hemisphere). Blackbody: $B(T,\nu)$, flux πB, $\int \pi B\,\mathrm{d}\nu = \sigma_\mathrm{R} T^4$. A Lambert surface looks equally bright from all directions because Lambert's cosine law and the projected solid angle cancel.

Temperatures. Energy balance $4\pi R_p^2 F_{\mathrm{S},p} = (1-A_\mathrm{B}) F_{\mathrm{S},*} (R_*/a)^2 \pi R_p^2 + L_\mathrm{int}$ gives $T_\mathrm{eq} = T_{\mathrm{eff},*} (R_*/a)^{1/2} [f'(1-A_\mathrm{B})]^{1/4}$ with $f' = 1/4$ (uniform) or $2/3$ (dayside). T_eff matches the total flux; $T_\mathrm{b}(\nu)$ matches the flux at one frequency and is the only measurable one.

Albedos. A_g compares the full-phase flux to a Lambert disk (Lambert sphere: $2/3$; can exceed 1). $A_\mathrm{S} = A_g q$ with the phase integral $q = 2\int \Phi_\alpha \sin\alpha\,\mathrm{d}\alpha$ ($3/2$ for Lambert); A_B is A_S averaged over the stellar spectrum and is what enters the energy balance. Only the apparent albedo is measurable from a single observation.

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